1.6 Beyond leading order

Coefficient functions and schemes, heavy quarks, and universality: the same distributions in Drell–Yan, jets and top production.

1. What this chapter adds

Chapter 1.4 derived the collinear logarithm from one gluon emission, absorbed it into the quark distribution, and obtained the DGLAP equation. It left three things open and asserted a fourth.

1. **The coupling.** Every formula carried a factor $\alpha_S(\mu^2)$, with the leading-order running quoted in Eq. (1.4.27). Section 2 derives the running from the QCD β-function at one and two loops and computes the coupling at the scale where our fits start.
2. **The finite parts.** The logarithm was regulated with a cut-off κ^2, and the remainder $C(x)$ was not computed. Sections 3 to 5 redo the calculation in dimensional regularisation, which is how the $\overline{\text{MS}}$ distributions of every modern fit are defined, and obtain the coefficient functions. Sections 6 and 7 show what changes, and what cannot, when the scheme or the scales are chosen differently.
3. **Quark masses.** Charm and bottom are neither light nor heavy on the scale of the data. Section 8 sets out the two limiting treatments and the matched scheme, FONLL, that our theory uses.
4. **Universality.** Chapter 1.4 stated that the same distributions enter every hard process. Sections 9 and 10 state the factorisation theorem for hadron collisions, derive the Drell–Yan cross-section at lowest order, show that at order α_S its collinear poles are the ones already absorbed in deep inelastic scattering (DIS), and list which process constrains which distribution.

The chapter follows Moore's thesis, sections 1.1.3 to 1.1.7 [1], for the dimensional-regularisation calculation, and Ellis, Stirling and Webber (ESW) [2] for the coupling, the coefficient functions and the Drell–Yan process. A parton distribution function is abbreviated PDF from here on.

2. The running coupling

The β-function

Loop corrections in QCD are ultraviolet divergent. Renormalisation absorbs the divergences into the coupling, at the price of a reference scale μ at which the coupling is defined. The coupling then depends on μ, and the dependence is fixed by the theory:

$$\begin{aligned} \mu^2 \frac{d\alpha_S}{d\mu^2} &= \beta(\alpha_S) = -b\,\alpha_S^2\left(1 + b'\alpha_S + \dots\right), \\ b &= \frac{33 - 2n_f}{12\pi}, \qquad b' = \frac{153 - 19n_f}{2\pi(33 - 2n_f)}. \end{aligned} \tag{1.6.1}$$

Here n_f is the number of quark flavours lighter than μ. The coefficients b and b' come from one-loop and two-loop diagrams and are quoted from ESW, chapter 2 [2]. In b, the $33 = 11C_A$ comes from gluon loops and the $-2n_f$ from quark loops. For $n_f \le 16$ the gluon term wins, b is positive, and the coupling falls as the scale rises. This is asymptotic freedom. It is the reason a quark struck at large Q^2 can be treated as free, and so the reason perturbation theory applies to hard scattering at all.

One and two loops

Keeping only the first term, Eq. (1.6.1) reads $d(1/\alpha_S)/d\ln\mu^2 = b$, which integrates at once:

$$\alpha_S(\mu^2) = \frac{\alpha_S(\mu_0^2)}{1 + b\,\alpha_S(\mu_0^2)\,\ln(\mu^2/\mu_0^2)}. \tag{1.6.2}$$

With the second term kept, the equation is still separable (derivation D1). The solution is implicit:

$$\frac{1}{\alpha_S(\mu^2)} - \frac{1}{\alpha_S(\mu_0^2)} + b' \ln\!\left[\frac{\alpha_S(\mu^2)}{1 + b'\alpha_S(\mu^2)} \cdot \frac{1 + b'\alpha_S(\mu_0^2)}{\alpha_S(\mu_0^2)}\right] = b\,\ln\frac{\mu^2}{\mu_0^2}. \tag{1.6.3}$$

Expanded in powers of $\alpha_S(\mu_0^2)$, Eq. (1.6.2) is $\alpha_S(\mu_0^2)\sum_n \big[-b\,\alpha_S(\mu_0^2)\ln(\mu^2/\mu_0^2)\big]^n$: the one-loop solution sums one logarithm for every power of the coupling. The two-loop solution also sums the terms with one logarithm fewer. This is the counting of the DGLAP table in chapter 1.4.

Λ

A first-order equation has one constant of integration. It can be given as the value of the coupling at a reference scale, or as the scale at which the one-loop coupling would diverge:

$$\begin{aligned} \alpha_S(\mu^2) &= \frac{1}{b\,L} && \text{(one loop)}, \\ \alpha_S(\mu^2) &= \frac{1}{b\,L}\left[1 - \frac{b'}{b}\,\frac{\ln L}{L} + \dots\right] && \text{(two loops)}, \end{aligned} \qquad L = \ln\frac{\mu^2}{\Lambda^2}. \tag{1.6.4}$$

Λ marks where perturbation theory fails. It is of the order of a few hundred MeV, the scale of hadron masses. Its value depends on the order, on n_f and, beyond one loop, on the renormalisation scheme and on how the constant is assigned in the expansion. For that reason the modern convention is to quote $\alpha_S(M_Z)$ instead.

Numbers for our fits

Our theory fixes $\alpha_S(M_Z) = 0.118$ and starts the evolution at $Q_0 = 1.65$ GeV. The check script runs the coupling down from $M_Z = 91.2$ GeV with $n_f = 5$ to the bottom mass, $m_b = 4.92$ GeV in the theory used here, and with $n_f = 4$ from there to Q_0, which lies above the charm mass $m_c = 1.51$ GeV. The coupling is continuous at m_b. These are the numbers of the worked example in chapter 1.5, section 5.

	$\alpha_S(m_b)$	$\alpha_S(Q_0)$	Λ FOR $n_f = 5$
one loop, Eq. (1.6.2)	0.204	0.289	88 MeV
two loops, Eq. (1.6.3)	0.213	0.327	226 MeV (expanded form of Eq. (1.6.4))

The one-loop Λ for $n_f = 4$ is 121 MeV. The same coupling at M_Z thus corresponds to values of Λ that differ by a factor of 2.6 between one and two loops.

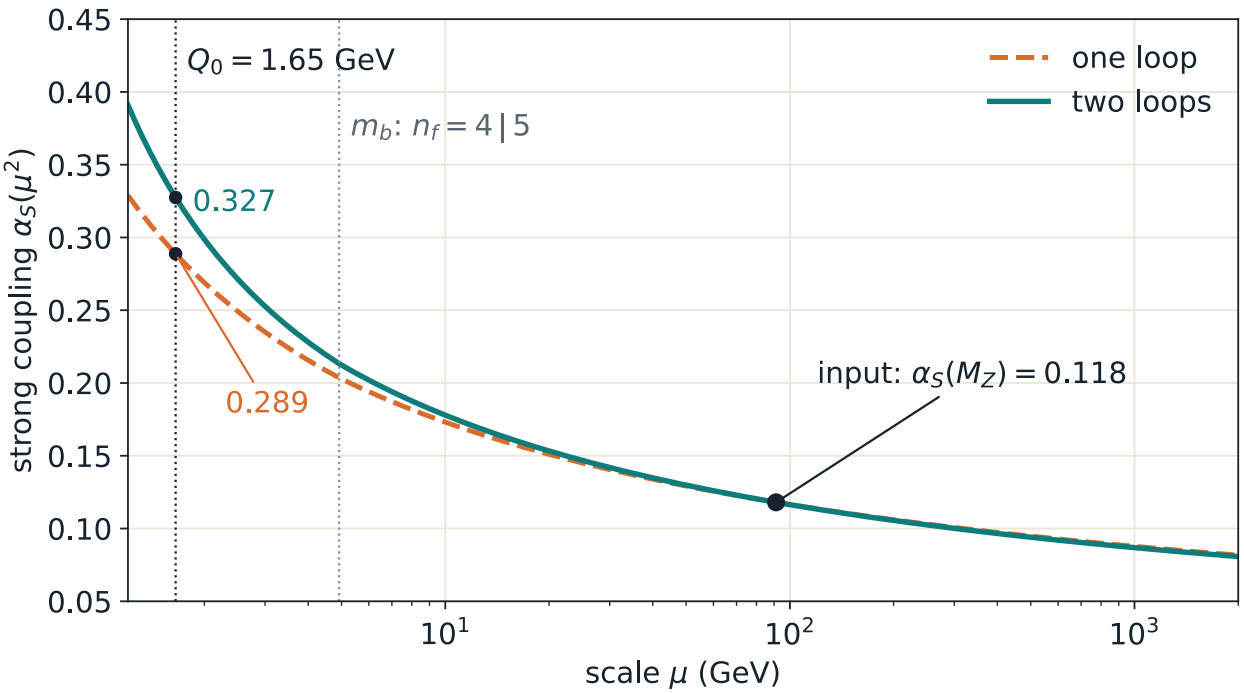

Figure 1. The coupling run down from $\alpha_S(M_Z) = 0.118$ with Eq. (1.6.2) (dashed) and Eq. (1.6.3) (solid), with $n_f = 5$ above m_b and $n_f = 4$ below. The two agree at high scales and separate by 13% at Q_0. Computed by the check script.

At Q_0 the expansion parameter α_S/π is about 0.10, and the coupling alone changes by 13% between one and two loops. The low-scale end of a fit is where the perturbative order matters most. This is one reason fits are done at next-to-next-to-leading order (NNLO).

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3. The quark structure function in d dimensions

This section sets up the order-α_S calculation of chapter 1.4 again, with the collinear divergence regulated by the number of space-time dimensions instead of a cut-off. The result is an exact expression, Eq. (1.6.10), whose singularities are extracted in section 4.

Why change the regulator

The cut-off κ^2 of chapter 1.4 isolates the logarithm but does not define the finite part. The constant left behind depends on what exactly is cut off: the virtuality, the transverse momentum, or a quark mass each give a different $C(x)$. Dimensional regularisation removes that ambiguity. Work in $d = 4 - 2\epsilon$ dimensions. An integral $\int dk_T^2/k_T^2$ becomes $\int dk_T^2/(k_T^2)^{1+\epsilon}$, which converges at its lower end when $\epsilon < 0$, that is, in more than four dimensions. The divergence reappears as a pole in ϵ. Ultraviolet divergences are regulated by the opposite sign of ϵ; in this calculation they cancel among the virtual diagrams, so only infrared poles are left and a single $\epsilon < 0$ serves.

In d dimensions the coupling g has a mass dimension. Write it as $g\,\mu^\epsilon$ with g dimensionless. The scale μ is arbitrary and has no physical meaning.

Two projections

Take a quark of momentum p as the target, with $z = Q^2/2p\cdot q$ its Bjorken variable. When the quark comes from a proton with momentum fraction ξ, $z = x/\xi$. The hadronic tensor of the quark has the general form

$$\hat{W}^{\alpha\beta} = \Big(-g^{\alpha\beta} + \frac{q^\alpha q^\beta}{q^2}\Big)\hat{F}_1 + \Big(p^\alpha - \frac{\nu}{q^2}q^\alpha\Big)\Big(p^\beta - \frac{\nu}{q^2}q^\beta\Big)\frac{\hat{F}_2}{\nu}, \qquad \nu = p\cdot q. \tag{1.6.5}$$

Contracting with the light-cone vector n of chapter 1.4 gives back $\hat{F}_2 = \nu\, n_\alpha n_\beta \hat{W}^{\alpha\beta}$, Eq. (1.4.4). In a covariant calculation two other contractions are more convenient. With $g_{\alpha\beta}g^{\alpha\beta} = d = 4-2\epsilon$, $p^2 = 0$ and $\hat{F}_L = \hat{F}_2 - 2z\hat{F}_1$:

$$-g_{\alpha\beta}\hat{W}^{\alpha\beta} = (1-\epsilon)\frac{\hat{F}_2}{z} - \frac{3-2\epsilon}{2}\,\frac{\hat{F}_L}{z}, \qquad p_\alpha p_\beta \hat{W}^{\alpha\beta} = \frac{Q^2}{8z^3}\,\hat{F}_L. \tag{1.6.6}$$

The second contraction isolates the longitudinal structure function. Define $\hat{T} \equiv -g_{\alpha\beta}\hat{W}^{\alpha\beta}/(1-\epsilon)$. Then

$$\frac{\hat{F}_2}{z} = \hat{T} + \frac{3-2\epsilon}{2(1-\epsilon)}\,\frac{\hat{F}_L}{z}.$$

At lowest order, $\gamma^* q \to q$, the trace gives $-g_{\alpha\beta}\overline{\sum}|\mathcal{M}|^2_{\alpha\beta} = 4(1-\epsilon)\,e_q^2\,\nu$. The factor $1-\epsilon$ counts the $d-2$ transverse photon polarisations. With the phase space and the $1/4\pi$ of Eq. (1.4.5), $\hat{T} = e_q^2\,\delta(1-z)$ and $\hat{F}_L = 0$, as in four dimensions.

Real emission: variables

Now add the gluon, $\gamma^*(q) + q(p) \to q(l) + g(r)$. In a covariant gauge both diagrams contribute: the one of Figure 1 of chapter 1.4, with the gluon radiated before the photon is absorbed, and the one with the gluon radiated from the outgoing quark. Describe the final state by two dimensionless variables. The first is z itself, which fixes the invariant mass of the final state. The second, v, is the angle of the gluon:

$$\hat{s} = (p+q)^2 = 2\,r\cdot l = Q^2\,\frac{1-z}{z}, \qquad 2\,r\cdot p = \frac{Q^2}{z}\,v, \qquad v = \frac{1-\cos\theta}{2}, \tag{1.6.7}$$

where θ is the angle between the gluon and the incoming quark in the centre-of-mass frame of the final state. Moore calls these variables u and v [1]. Two limits matter.

- $v \to 0$: the gluon is parallel to the incoming quark. The propagator of Figure 1 of chapter 1.4, $1/(p-r)^2 = -1/(2r\cdot p)$, diverges. This is the **collinear** limit, the $k_T \to 0$ region of chapter 1.4.
- $z \to 1$: the final state has no invariant mass. The gluon carries no energy, or travels along the outgoing quark. The propagator of the other diagram, $1/(l+r)^2 = 1/\hat{s}$, diverges. This is the **soft** limit, the $x \to 1$ singularity of $\hat{P}(x)$ in chapter 1.4.

Phase space and matrix element

In d dimensions the two-particle phase space, integrated over all angles except θ, is (derivation D2)

$$\int d\Phi_2 = \frac{1}{8\pi}\,\frac{1}{\Gamma(1-\epsilon)}\Big(\frac{4\pi}{\hat{s}}\Big)^\epsilon \int_0^1 dv\,\big[v(1-v)\big]^{-\epsilon}. \tag{1.6.8}$$

At $\epsilon = 0$ it reduces to the familiar $1/8\pi$. The factor $[v(1-v)]^{-\epsilon}$ is $(\sin\theta)^{-2\epsilon}$: in more than four dimensions there is less phase space near the beam axis, and that is what tames the collinear divergence.

The matrix element, summed over both diagrams and over gluon polarisations and averaged over the spin and colour of the incoming quark, is quoted from Moore, Eq. (1.59) [1]:

$$-g_{\alpha\beta}\overline{\sum}|\mathcal{M}|^2_{\alpha\beta} = 4e_q^2 g^2 \mu^{2\epsilon} C_F (1-\epsilon)\left[(1-\epsilon)\Big(\frac{v}{1-z}+\frac{1-z}{v}\Big)+\frac{2z(1-v)}{v(1-z)}+2\epsilon\right]. \tag{1.6.9}$$

The check script evaluates the left side at $\epsilon = 0$ with explicit Dirac matrices and reproduces the bracket to one part in 10^8. The terms proportional to ϵ are not tested that way and rest on the source. The singularities are visible directly: $1/v$ is collinear, $1/(1-z)$ is soft, and the third term has both.

Putting Eqs. (1.6.8) and (1.6.9) together with the factor $1/4\pi$, and using $\hat{s} = Q^2(1-z)/z$ and $g^2 = 4\pi\alpha_S$:

$$\begin{aligned}\hat{T}^{\,\mathrm{real}}(z) = e_q^2\,\frac{\alpha_S}{2\pi}\,C_F\,\frac{1}{\Gamma(1-\epsilon)}\Big(\frac{4\pi\mu^2}{Q^2}\Big)^{\epsilon}\frac{z^{\epsilon}}{(1-z)^{\epsilon}}\int_0^1 dv\,\big[v(1-v)\big]^{-\epsilon} \\ \times\left[(1-\epsilon)\Big(\frac{v}{1-z}+\frac{1-z}{v}\Big)+\frac{2z(1-v)}{v(1-z)}+2\epsilon\right].\end{aligned} \tag{1.6.10}$$

Given Eq. (1.6.9), no approximation has been made. Chapter 1.4 kept only the logarithm; this expression contains the complete order-α_S real emission.

4. Soft and collinear poles

This section extracts the singularities of Eq. (1.6.10). It establishes three results: the angular integral produces a single pole, the soft limit turns it into a double pole, and the double pole cancels against the virtual diagrams while one collinear pole survives, multiplied by the splitting function.

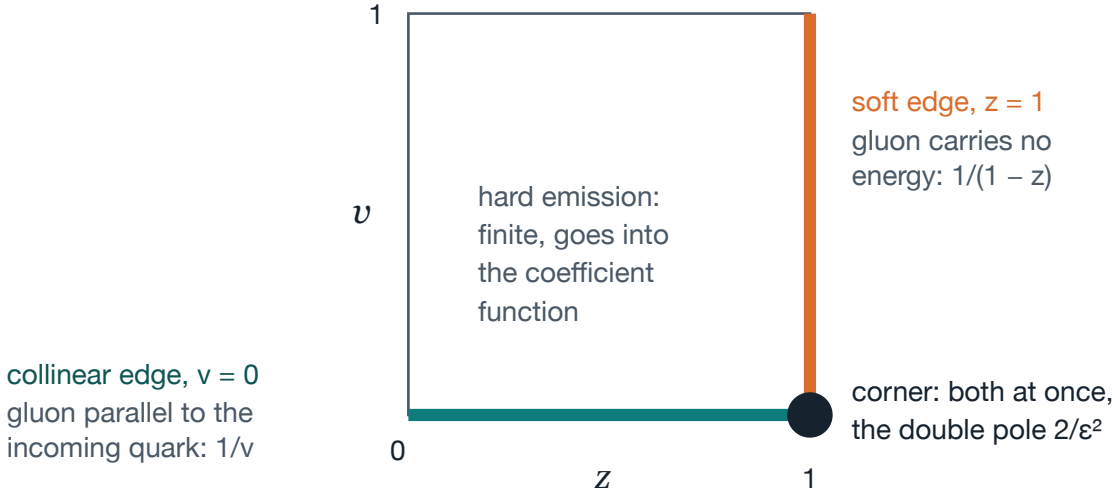

Figure 2. The phase space of the emitted gluon in the variables of Eq. (1.6.7). The matrix element is singular on two edges. Integrating over v turns the collinear edge into a pole $1/\epsilon$ at every z. Integrating over z against a smooth function turns the soft edge into a second pole, which exists only at $z = 1$. The corner gives $2/\epsilon^2$ and cancels against the virtual diagrams. The collinear edge away from the corner does not cancel.

The angular integral

Every term in Eq. (1.6.10) is a power of v times a power of $1-v$, so the v integral is a sum of Euler Beta functions. The result (derivation D4) is

$$\hat{T}^{\,\mathrm{real}}(z) = e_q^2\,\frac{\alpha_S}{2\pi}\,C_F\,c_\epsilon\,\frac{z^\epsilon}{(1-z)^{1+\epsilon}}\,N(z,\epsilon), \qquad c_\epsilon = \Big(\frac{4\pi\mu^2}{Q^2}\Big)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}, \tag{1.6.11}$$

$$N(z,\epsilon) = -\frac{(1-\epsilon)(1-z)^2}{\epsilon} + \frac{1-\epsilon}{2(1-2\epsilon)} - \frac{2z(1-\epsilon)}{\epsilon(1-2\epsilon)} + \frac{2\epsilon(1-z)}{1-2\epsilon}.$$

The terms in $1/\epsilon$ come from $\int_0^1 dv\, v^{-1-\epsilon} = -1/\epsilon$. This is the collinear pole. For small ϵ,

$$N(z,\epsilon) = -\frac{1+z^2}{\epsilon} + \Big(z^2 - 4z + \frac{3}{2}\Big) + \epsilon\Big(\frac{5}{2} - 6z\Big) + O(\epsilon^2). \tag{1.6.12}$$

The coefficient of the pole is $-(1+z^2)$. Together with the $1/(1-z)$ in Eq. (1.6.11) it is the unregularised splitting function $\hat{P}(z)$ of Eq. (1.4.8). The collinear singularity again comes with the splitting function, now as the residue of a pole instead of the coefficient of a logarithm.

The soft limit as a distribution

The factor $z^\epsilon(1-z)^{-1-\epsilon}$ cannot be expanded in ϵ term by term, because $1/(1-z)$ is not integrable at $z=1$. It has to be expanded as a distribution, that is, under an integral with a smooth function. The result (derivation D3) is

$$\frac{z^\epsilon}{(1-z)^{1+\epsilon}} = -\frac{1}{\epsilon}\,\delta(1-z) + \frac{1}{(1-z)_+} - \epsilon\left[\Big(\frac{\ln(1-z)}{1-z}\Big)_+ - \frac{\ln z}{1-z}\right] + O(\epsilon^2). \tag{1.6.13}$$

The plus distributions are defined as in Eq. (1.4.13): $\int_0^1 dz\,[h(z)]_+ f(z) = \int_0^1 dz\, h(z)\,[f(z)-f(1)]$. The first term is the soft pole. It sits at $z=1$ only, because a gluon with no energy does not change the quark's momentum.

Multiply Eq. (1.6.13) by Eq. (1.6.12) and keep everything that survives as $\epsilon \to 0$. At $z=1$ the three coefficients of N are -2, $-\frac{3}{2}$ and $-\frac{7}{2}$, so the delta-function terms are $2/\epsilon^2 + 3/2\epsilon + 7/2$:

$$\begin{aligned}\frac{z^\epsilon N(z,\epsilon)}{(1-z)^{1+\epsilon}} = {}& \Big(\frac{2}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{7}{2}\Big)\delta(1-z) - \frac{1}{\epsilon}\,\frac{1+z^2}{(1-z)_+} \\ & + (1+z^2)\Big(\frac{\ln(1-z)}{1-z}\Big)_+ - \frac{1+z^2}{1-z}\ln z \\ & - \frac{3}{2}\,\frac{1}{(1-z)_+} + 3 - z + O(\epsilon).\end{aligned} \tag{1.6.14}$$

The double pole is the product of the soft pole in Eq. (1.6.13) and the collinear pole in Eq. (1.6.12). It comes from the corner of Figure 2.

Virtual diagrams and the cancellation

The virtual diagrams have the lowest-order final state, so they contribute at $z=1$ only. Their sum is quoted from Moore, Eq. (1.70) [1]:

$$\hat{T}^{\,\mathrm{virtual}}(z) = -e_q^2\,\frac{\alpha_S}{2\pi}\,C_F\,c_\epsilon\Big(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8 + \frac{\pi^2}{3}\Big)\delta(1-z). \tag{1.6.15}$$

Add Eqs. (1.6.14) and (1.6.15). The double poles cancel. The single poles combine as $-\frac{1}{\epsilon}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]$, which is $-P_{qq}(z)/(\epsilon\, C_F)$ with the complete splitting function of Eq. (1.4.12). Call the finite remainder $R(z)$:

$$\begin{aligned}\hat{T}(z) &= e_q^2\left\{\delta(1-z) + \frac{\alpha_S}{2\pi}\,c_\epsilon\left[-\frac{1}{\epsilon}\,P_{qq}(z) + R(z)\right]\right\},\\ R(z) &= C_F\left[(1+z^2)\Big(\frac{\ln(1-z)}{1-z}\Big)_+ - \frac{3}{2}\frac{1}{(1-z)_+} - \frac{1+z^2}{1-z}\ln z\right.\\ &\qquad\left. + 3 - z - \Big(\frac{9}{2} + \frac{\pi^2}{3}\Big)\delta(1-z)\right].\end{aligned} \tag{1.6.16}$$

Three points about this result.

- **The soft singularity has cancelled completely**, as chapter 1.4 argued from quark-number conservation. Here the cancellation is explicit: $2/\epsilon^2$ from real emission against $-2/\epsilon^2$ from the loops.
- **The coefficient $\frac{3}{2}$ of $\delta(1-z)$ in P_{qq} has been computed, not inferred.** Chapter 1.4 fixed it by requiring that the splitting function integrate to zero. Here real emission contributes $+\frac{3}{2\epsilon}$ and the loops $-\frac{3}{\epsilon}$, and the sum is $-\frac{1}{\epsilon}\cdot\frac{3}{2}$.
- **One pole is left**, and it is collinear. Its residue is the splitting function. Nothing in the partonic calculation can cancel it.

The finite delta-function coefficient, $\frac{7}{2} - 8 - \frac{\pi^2}{3} = -\left(\frac{9}{2} + \frac{\pi^2}{3}\right)$, can be tested independently. Quark number is conserved, so the finite order-α_S correction to $\hat{F}_2/z$ must integrate to zero over z once the F_L term below is added. The check script confirms that it does: the loop result and the conservation law agree.

The longitudinal projection

The second contraction in Eq. (1.6.6) is finite. From the same two diagrams, $p_\alpha p_\beta \overline{\sum}|\mathcal{M}|^2_{\alpha\beta} = 2e_q^2 g^2 C_F\, Q^2(1-v)/z$ at $\epsilon = 0$ (Moore, Eq. (1.60); checked with explicit Dirac matrices). There is no $1/v$ and no $1/(1-z)$, so four dimensions suffice. With $\int d\Phi_2 = \frac{1}{8\pi}\int_0^1 dv$ and $\int_0^1 (1-v)\,dv = \frac{1}{2}$,

$$p_\alpha p_\beta \hat{W}^{\alpha\beta} = \frac{1}{4\pi}\cdot\frac{1}{8\pi}\cdot 2e_q^2\,(4\pi\alpha_S)\,C_F\,\frac{Q^2}{z}\cdot\frac{1}{2} = e_q^2\,\frac{\alpha_S C_F}{8\pi}\,\frac{Q^2}{z}.$$

By the second relation of Eq. (1.6.6), $\hat{F}_L = (8z^3/Q^2)\,p_\alpha p_\beta \hat{W}^{\alpha\beta}$:

$$\hat{F}_L(z) = e_q^2\,\frac{\alpha_S}{2\pi}\,2C_F\,z^2. \tag{1.6.17}$$

A spin-½ quark cannot absorb a longitudinal photon at lowest order, which is the Callan–Gross relation. Once it has radiated a gluon and acquired transverse momentum it can. F_L therefore starts at order α_S, with no logarithm and no divergence.

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5. The $\overline{\text{MS}}$ distributions and the coefficient functions

This section turns Eq. (1.6.16) into a definition of the quark distribution and reads off the coefficient functions that chapter 1.4 left unspecified.

The definition

Fold the quark's $\hat{T}$ with a bare distribution $q_0(\xi)$, as in Eq. (1.4.14). The pole multiplies c_ϵ, and

$$-\frac{c_\epsilon}{\epsilon} = -\frac{1}{\epsilon} - \ln 4\pi + \gamma_E + \ln\frac{Q^2}{\mu^2} + O(\epsilon),$$

where $\gamma_E = 0.5772\ldots$ is Euler's constant, from $\Gamma(1-\epsilon)/\Gamma(1-2\epsilon) = 1 - \gamma_E\,\epsilon + \ldots$. The constants $\ln 4\pi - \gamma_E$ are an artefact of continuing the phase space to d dimensions and appear in every such calculation alongside $1/\epsilon$. The $\overline{\text{MS}}$ (modified minimal subtraction) scheme absorbs the pole and exactly these constants into the distribution, and nothing else. Including the gluon-initiated term, which has the same structure with P_{qg}:

$$q(x,\mu_F^2) = q_0(x) + \frac{\alpha_S}{2\pi}\Big(-\frac{1}{\epsilon} - \ln 4\pi + \gamma_E + \ln\frac{\mu_F^2}{\mu^2}\Big)\Big[P_{qq}\otimes q_0 + P_{qg}\otimes g_0\Big](x). \tag{1.6.18}$$

This is Moore's Eq. (1.73) and ESW's Eq. (9.22) [1, 2]. Here μ_F is the factorisation scale, written μ in chapters 1.3 to 1.5. The subscript is needed from here on, because μ now denotes the mass unit of dimensional regularisation (called μ_d in chapter 1.3) and a second scale μ_R enters in section 7. Compare it with the cut-off definition, Eq. (1.4.15): $-1/\epsilon$ has taken the place of $\ln(\mu_F^2/\kappa^2)$, and the finite function C is no longer absorbed. The bare distribution q_0 must itself diverge as $\epsilon \to 0$ so that $q(x,\mu_F^2)$ is finite; it was never observable. The arbitrary scale μ of dimensional regularisation cancels between the two.

Because $-1/\epsilon$ is positive for $\epsilon < 0$, the subtraction has the same sign as the logarithm it replaces. Taking $\mu_F^2\,\partial/\partial\mu_F^2$ of Eq. (1.6.18) gives the DGLAP equation (1.4.18) directly, with the gluon term of Eq. (1.4.23).

The definition contains only the splitting functions. Nothing in it refers to the photon or to deep inelastic scattering. This is the root of universality, taken up in section 9.

The coefficient functions for F_2

Substituting Eq. (1.6.18) into the folded $\hat{T}$, and adding $\frac{3}{2}\hat{F}_L/z$ from Eq. (1.6.17) as Eq. (1.6.6) requires, gives the structure function to order α_S:

$$\begin{aligned} F_2(x,Q^2) = x\sum_{q,\bar q} e_q^2 \int_x^1 \frac{d\xi}{\xi}\,\Bigg[& q(\xi,\mu_F^2)\Big\{\delta(1-z) + \frac{\alpha_S}{2\pi}\Big(P_{qq}(z)\ln\frac{Q^2}{\mu_F^2} + C_q(z)\Big)\Big\} \\ & + g(\xi,\mu_F^2)\,\frac{\alpha_S}{2\pi}\Big(P_{qg}(z)\ln\frac{Q^2}{\mu_F^2} + C_g(z)\Big)\Bigg]_{z=x/\xi}. \end{aligned} \tag{1.6.19}$$

The longitudinal term adds $3z\,C_F$ to the $3 - z$ of Eq. (1.6.16). The quark coefficient function in the $\overline{\text{MS}}$ scheme is therefore

$$\begin{aligned} C_q(z) = C_F\Bigg[& 2\Big(\frac{\ln(1-z)}{1-z}\Big)_+ - \frac{3}{2}\frac{1}{(1-z)_+} - (1+z)\ln(1-z) \\ & - \frac{1+z^2}{1-z}\ln z + 3 + 2z - \Big(\frac{\pi^2}{3} + \frac{9}{2}\Big)\delta(1-z)\Bigg], \end{aligned} \tag{1.6.20}$$

where $(1+z^2)\big(\frac{\ln(1-z)}{1-z}\big)_+ = 2\big(\frac{\ln(1-z)}{1-z}\big)_+ - (1+z)\ln(1-z)$ has been used. This is ESW's Eq. (4.85) [2], and it is equal as a distribution to the compact form in Moore's Eq. (1.79), which the check script verifies.

The gluon coefficient function comes from $\gamma^* g \to q\bar q$, treated the same way. Its pole is $-P_{qg}/\epsilon$ and is removed by the second term of Eq. (1.6.18). The finite part is quoted from ESW, Eq. (4.85):

$$C_g(z) = T_R\Big[\big(z^2 + (1-z)^2\big)\ln\frac{1-z}{z} - 1 + 8z(1-z)\Big]. \tag{1.6.21}$$

In Eq. (1.6.19) it is summed over quarks and antiquarks, like P_{qg} in Eq. (1.4.20). Section 8 obtains the same function a second way, as the massless limit of a calculation with a quark mass.

The coefficient functions for F_L

From Eq. (1.6.17), and the corresponding gluon term quoted from ESW, Eq. (4.86):

$$F_L(x,Q^2) = x\sum_{q,\bar q} e_q^2\,\frac{\alpha_S}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[2C_F\,z\;q(\xi,Q^2) + 4T_R\,z(1-z)\;g(\xi,Q^2)\right]_{z=x/\xi}. \tag{1.6.22}$$

These have no pole, so nothing is subtracted and they are the same in every scheme at this order. F_L is the one DIS observable to which the gluon contributes at the leading non-vanishing order.

Size and shape

Three features of Eq. (1.6.20) are worth keeping in mind.

- **It integrates to zero.** $\int_0^1 C_q(z)\,dz = 0$, so a sum rule that counts quarks through $\int dx\,F_2/x$ of a non-singlet combination has no order-α_S correction.
- **Its second moment is small.** $\int_0^1 z\,C_q(z)\,dz = C_F/6 = 2/9$ and $\int_0^1 z\,C_g(z)\,dz = -T_R/4$. At Q_0, with the two-loop coupling of section 2, $\frac{\alpha_S}{2\pi}\cdot\frac{2}{9} = 0.012$. Averaged over x, the correction is at the per-cent level.
- **It is large near $z=1$.** The term $2C_F\big(\frac{\ln(1-z)}{1-z}\big)_+$ is what remains of the soft double pole after the cancellation. At large x the convolution probes z close to 1 and the correction grows like $\alpha_S\ln^2(1-x)$. Such threshold logarithms recur at every order (chapter 3.3).

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6. The DIS scheme, and changing scheme

This section shows that the choice of scheme moves a finite convolution between the distributions and the coefficient functions, and that no prediction changes at the order to which one is working.

The transformation

The DIS scheme of chapter 1.4 absorbs the whole of the order-α_S correction to F_2 into the quark distributions. Comparing with Eq. (1.6.19) at $\mu_F^2 = Q^2$, for each quark and antiquark flavour,

$$\begin{aligned} q^{\rm DIS} &= q + \frac{\alpha_S}{2\pi}\big[C_q\otimes q + C_g\otimes g\big], \\ g^{\rm DIS} &= g - \frac{\alpha_S}{2\pi}\sum_{q,\bar q}\big[C_q\otimes q + C_g\otimes g\big], \end{aligned} \tag{1.6.23}$$

where unlabelled distributions are in $\overline{\rm MS}$ and all are at the same scale. The first line makes $F_2 = x\sum e_q^2\,q^{\rm DIS}$ exact. The photon does not couple to the gluon, so F_2 says nothing about how to define $g^{\rm DIS}$. The second line is a convention. It is chosen so that the total momentum $\int_0^1 x\,(\Sigma+g)\,dx$ is the same in both schemes, and the same form is then used at all x.

Predictions do not change

Take any other observable with a perturbative expansion in $\overline{\rm MS}$, written for brevity with one quark distribution: $\sigma = \big(h_0 + \frac{\alpha_S}{2\pi}h_1\big)\otimes q$. Invert Eq. (1.6.23) to first order, $q = q^{\rm DIS} - \frac{\alpha_S}{2\pi}C_q\otimes q^{\rm DIS}$, and substitute:

$$\sigma = \Big(h_0 + \frac{\alpha_S}{2\pi}\big[h_1 - h_0 \otimes C_q\big]\Big) \otimes q^{\mathrm{DIS}} + O(\alpha_S^2).$$

The prediction is the same to order α_S, with a different coefficient function, $h_1^{\mathrm{DIS}} = h_1 - h_0 \otimes C_q$. The two schemes differ at order α_S^2, which is beyond the accuracy of either. The check script confirms this numerically for F_L in moment space: the two schemes' predictions differ by an amount that falls by a factor of 4 when α_S is halved.

The evolution equations change in the same controlled way. For a non-singlet moment $q(N)$, differentiating $q^{\mathrm{DIS}} = \big(1 + \frac{\alpha_S}{2\pi} C_q(N)\big) q$ and using Eq. (1.6.1) gives (derivation D5)

$$\gamma_{\mathrm{NS}}^{(1),\mathrm{DIS}}(N) = \gamma_{\mathrm{NS}}^{(1),\overline{\mathrm{MS}}}(N) - 2\pi b\, C_q(N). \tag{1.6.24}$$

Here $\gamma^{(1)}$ is the moment of the next-to-leading-order splitting function $P^{(1)}$ of Eq. (1.4.34). The leading-order splitting functions are the same in every scheme. Scheme dependence starts at next-to-leading order, in the coefficient functions and in $P^{(1)}$ together, and cancels between them.

> **What a scheme is.** A parton distribution beyond leading order is not a physical quantity. It is defined by a subtraction, and a different subtraction gives a different function. What is physical is the combination of a distribution with the coefficient function computed in the same scheme. A set of PDFs must always be used with cross-sections in the scheme, and at the perturbative order, in which it was fitted.

Why $\overline{\mathrm{MS}}$

The DIS scheme is tied to one observable, and its gluon is a convention. The $\overline{\mathrm{MS}}$ definition, Eq. (1.6.18), refers to no process, treats quarks and gluons alike, and extends to any order. It is the scheme in which the hard cross-sections of section 10 are calculated. Moore's thesis considers no other [1], and our fits use it throughout.

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7. Two scales, and what varying them tells you

This section explains why a prediction truncated at a finite order depends on two arbitrary scales, and what can and cannot be learned from that dependence.

The renormalisation scale

Let R be an observable that depends on one hard scale Q and starts at order α_S^n. The coupling in its expansion can be defined at any scale μ_R. Changing μ_R changes the coupling according to Eq. (1.6.1), and the coefficients must change to compensate:

$$R = \alpha_S^n(\mu_R^2)\Big[r_0 + \alpha_S(\mu_R^2)\Big(r_1 + n\,b\,r_0 \ln\frac{\mu_R^2}{Q^2}\Big) + O(\alpha_S^2)\Big]. \tag{1.6.25}$$

Differentiating, the leading term gives $n\,\alpha_S^{n-1}(-b\,\alpha_S^2)\,r_0$ and the explicit logarithm gives $+n\,b\,r_0\,\alpha_S^{n+1}$. They cancel:

$$\mu_R^2 \frac{dR}{d\mu_R^2} = O\big(\alpha_S^{n+2}\big).$$

The all-orders sum does not depend on μ_R. A series truncated after k terms depends on it at the order of the first term left out.

The factorisation scale

Eq. (1.6.19) has the same structure. The distributions are evaluated at μ_F, and the coefficient function carries the compensating $P \ln(Q^2/\mu_F^2)$. By the DGLAP equation the derivative of the distribution cancels the derivative of the logarithm, and $dF_2/d\ln\mu_F^2 = O(\alpha_S^2)$. This is the argument of section 6 of chapter 1.4 read in reverse.

The two scales have different origins. μ_R is where the ultraviolet divergences are subtracted, and μ_F is where the collinear ones are. They are often set equal, but they are independent.

Scale variation

Since the dependence on μ_R and μ_F is of the order of the missing terms, its size is used as an estimate of them. The convention is to vary each scale by a factor of 2 either side of the hard scale.

A worked case, computed by the check script: the QCD correction to the rate of $e^+e^- \to$ hadrons at $Q = M_Z$, for which the series in $a = \alpha_S/\pi$ is $a + 1.411\,a^2 + \ldots$ with five flavours [2]. The coefficient is ESW's rounded $1.986 - 0.115\,n_f$; to four figures it is 1.410, which shifts the second-order entries below by 2×10^{-6}. In the notation of Eq. (1.6.25), $n = 1$.

	$\mu_R = M_Z/2$	$\mu_R = M_Z$	$\mu_R = 2M_Z$	SPREAD
first order	0.0420	0.0376	0.0340	0.0079
second order	0.0398	0.0396	0.0387	0.0010

The spread falls by a factor of about 8 from one order to the next, and the second-order result lies inside the first-order range. Here the estimate works.

Its limits

Scale variation has four weaknesses.

- **It only sees what the renormalisation group already knows.** The variation reproduces the logarithmic terms of the next order, such as $n\,b\,r_0 \ln(\mu_R^2/Q^2)$ in Eq. (1.6.25). The genuinely new coefficient of the next order is not constrained.
- **It misses new channels.** At next-to-leading order the Drell–Yan process acquires a quark–gluon initial state (section 9) that is absent at lowest order. No variation of scales in the lowest-order result can anticipate it.
- **The factor of 2 is a convention.** The resulting band has no statistical meaning. It is not a standard deviation.
- **In a fit, correlations matter.** The missing higher orders are different for each process but related between neighbouring data points. A fit needs their covariance, not only their size.

The Snowmass review states that theory uncertainties are important even in NNLO fits [3]. How they are estimated and included in a fit is the subject of chapter 3.1.

8. Heavy quarks

This section explains how a quark with a mass comparable to Q enters a structure function, and writes down the matched scheme used in our theory.

The problem

The splitting functions and coefficient functions so far assume massless quarks. For up, down and strange that is accurate at every scale where perturbation theory applies. For charm, with $m_c = 1.51$ GeV in our theory, and for bottom it is not. Much of the DIS data has Q within a factor of a few of m_c. There are two limiting treatments.

Fixed flavour number. Treat the proton as made of gluons and three light flavours only. Charm appears in the final state alone, produced in pairs by the hard scattering. At lowest order the process is $\gamma^* g \to c\bar{c}$, photon–gluon fusion [4]:

$$F_2^c(x,Q^2) = 2e_c^2\, x\, \frac{\alpha_S}{2\pi} \int_x^{1/(1+4\lambda)} \frac{dz}{z}\, C_g^{\,m}(z,\lambda)\, g\Big(\frac{x}{z},\mu^2\Big), \qquad \lambda = \frac{m_c^2}{Q^2}. \tag{1.6.26}$$

$$\begin{aligned} C_g^{\,m}(z,\lambda) = T_R\Big[&\big(z^2 + (1-z)^2 + 4\lambda z(1-3z) - 8\lambda^2 z^2\big)\ln\frac{1+\beta_c}{1-\beta_c} \\ &+ \big(8z(1-z) - 1 - 4\lambda z(1-z)\big)\,\beta_c\Big], \qquad \beta_c = \sqrt{1 - \frac{4\lambda z}{1-z}}. \end{aligned}$$

The factor 2 counts the quark and the antiquark. β_c is the velocity of the charm quark in the photon–gluon centre-of-mass frame. The upper limit of the integral is the threshold $\hat{s} \geq 4m_c^2$. The check script reproduces $C_g^{\,m}$ at four kinematic points from explicit Dirac traces and a numerical angular integral.

This treatment is exact in the mass at each order, and correct near threshold. It has no collinear divergence, because the mass cuts off the quark propagator.

Zero mass, variable flavour number. Above $\mu = m_c$, treat charm as a fourth massless parton with its own distribution $c(x,\mu^2)$, generated by evolution with $n_f = 4$. The structure function is given by Eq. (1.6.19) with charm included in the sum. This is simple, and at $Q \gg m_c$ it is the right description.

How the two are related

Take $\lambda \to 0$ in $C_g^{\,m}$. The velocity $\beta_c \to 1$ and the logarithm becomes large (derivation D6):

$$C_g^{\,m}(z,\lambda) \;\xrightarrow{\lambda\to 0}\; P_{qg}(z)\,\ln\frac{Q^2}{m_c^2} + C_g(z). \tag{1.6.27}$$

The mass has done the work of the cut-off κ^2 in Eq. (1.4.20). The logarithm multiplies the same splitting function, and the constant is the $\overline{\text{MS}}$ coefficient function of Eq. (1.6.21).

This exposes the weakness of each limit.

- The fixed-flavour result contains $\alpha_S \ln(Q^2/m_c^2)$, and at order n it contains $\alpha_S^n \ln^n(Q^2/m_c^2)$. At large Q^2 these are not small, and nothing sums them.
- The zero-mass result sums exactly these logarithms, through the evolution of $c(x,\mu^2)$. But it discards every term suppressed by m_c^2/Q^2, including the threshold. Near $Q \sim m_c$ it is wrong.

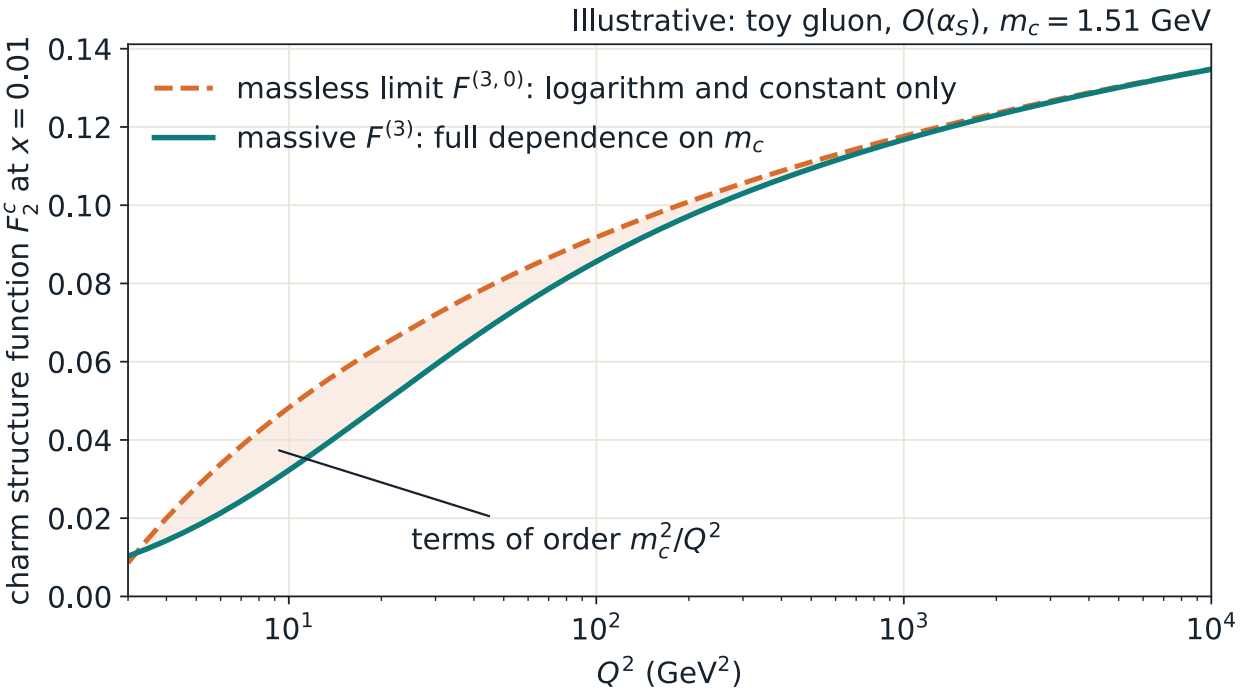

Figure 3. Photon–gluon fusion at order α_S with the full mass dependence, Eq. (1.6.26), and its massless limit, Eq. (1.6.27), folded with the same gluon. The shaded gap is the set of terms of order m_c^2/Q^2 that a zero-mass scheme drops. The ratio of the two curves is 0.64 at $Q^2 = 5$ GeV^2, 0.93 at 100 GeV^2 and 1.00 at 10^4 GeV^2. The gluon is a fixed toy shape, $x\,g = 1.7\,x^{-0.2}(1-x)^5$, and the curves are illustrative, not a fit.

General-mass schemes and FONLL

A general-mass variable-flavour-number scheme keeps both: the mass dependence near threshold and the summed logarithms far above it. Several constructions exist and are used in global fits for DIS [3]. Our theory uses FONLL [5]. Its definition is a matching formula:

$$F^{\mathrm{FONLL}}(x,Q^2) = F^{(3)}(x,Q^2) + F^{(4)}(x,Q^2) - F^{(3,0)}(x,Q^2). \tag{1.6.28}$$

Each term has a precise meaning.

- $F^{(3)}$ is the massive, fixed-flavour result: three light flavours, charm produced in the final state, coefficient functions with their full dependence on m_c, to a fixed order in α_S. Eq. (1.6.26) is its first term.
- $F^{(4)}$ is the massless result: four flavours, a charm distribution, logarithms of Q^2/m_c^2 summed by evolution, no powers of m_c^2/Q^2.
- $F^{(3,0)}$ is the massless limit of $F^{(3)}$: the same fixed-order expression with every term that vanishes as $m_c/Q \to 0$ dropped and the logarithms kept. Eq. (1.6.27) is its first term. These are the contributions present in both of the other terms. Subtracting them removes the double counting.

All three are written in terms of the same four-flavour distributions and coupling, which requires re-expressing $F^{(3)}$ using the relations between three-flavour and four-flavour quantities at the threshold.

The formula has the right limits. For $Q \gg m_c$, $F^{(3)} \to F^{(3,0)}$ and $F^{\mathrm{FONLL}} \to F^{(4)}$. Near threshold, the difference $F^{(4)} - F^{(3,0)}$ is of higher order in α_S than the massive calculation, so $F^{\mathrm{FONLL}} \to F^{(3)}$ up to terms beyond its accuracy. This can be seen at first order. Solving the leading-order evolution equation for a charm distribution that vanishes at

$\mu = m_c$, over a range short enough that the gluon does not change, gives the radiatively generated charm of chapter 1.5, section 7:

$$c(x,\mu^2) = \bar{c}(x,\mu^2) = \frac{\alpha_S}{2\pi}\,\ln\frac{\mu^2}{m_c^2}\,\big(P_{qg}\otimes g\big)(x) + O(\alpha_S^2). \tag{1.6.29}$$

Insert this in Eq. (1.6.19) at $\mu_F^2 = Q^2$. The charm terms of $F^{(4)}$ are $x\,e_c^2\big[c+\bar{c}+2\frac{\alpha_S}{2\pi}C_g\otimes g\big] = 2e_c^2\,x\,\frac{\alpha_S}{2\pi}\big[P_{qg}\ln\frac{Q^2}{m_c^2} + C_g\big]\otimes g$, which is $F^{(3,0)}$ exactly. At order α_S the last two terms of Eq. (1.6.28) cancel and FONLL reduces to the massive result. They differ from order α_S^2 on, where evolution starts to sum logarithms the fixed-order result does not have.

FONLL comes in variants according to the orders combined [5]. FONLL-A joins the order-α_S massive result to next-to-leading-order massless evolution and coefficients. FONLL-B uses the order-α_S^2 massive result with the same massless part. FONLL-C uses the order-α_S^2 massive result with NNLO massless evolution and coefficients. Our theory, 40000000, uses FONLL-C, which is also the scheme in Moore's fits [1]. Bottom is treated the same way at its own threshold.

For hadron-collider observables the situation is simpler and less complete. Nearly all of them are computed in fits with massless charm and bottom, which is justified by the high scales involved [3].

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9. Universality: from DIS to hadron collisions

This section states the factorisation theorem for collisions of two hadrons and works through its simplest case, the Drell–Yan process, at lowest order in full and at order α_S as far as the pole structure.

The theorem

For a hard process in the collision of hadrons with momenta P_1 and P_2, producing a final state with a large scale Q,

$$\begin{aligned}\sigma = \sum_{i,j}\int_0^1 dx_1\int_0^1 dx_2\; f_i(x_1,\mu_F^2)\,f_j(x_2,\mu_F^2) \\ \times\,\hat{\sigma}_{ij}\big(x_1P_1, x_2P_2;\,\alpha_S(\mu_R^2),\mu_F^2,\mu_R^2\big) + O\Big(\frac{\Lambda^2}{Q^2}\Big).\end{aligned} \tag{1.6.30}$$

The sum runs over quarks, antiquarks and the gluon. f_i is the distribution of parton i in the hadron, and $\hat{\sigma}_{ij}$ is the cross-section for partons i and j to produce the final state, calculated in perturbation theory.

The statement has two parts. The first is that the cross-section factorises at all: each hadron supplies one parton, and whatever else passes between the hadrons does not affect an inclusive cross-section at leading power. The second is that the f_i are the functions that appear in deep inelastic scattering. The second part follows from the form of Eq. (1.6.18): the subtraction that defines an $\overline{\text{MS}}$ distribution involves only the splitting functions, which describe the collinear radiation of a parton whatever it goes on to do [1]. The first part is harder. Soft gluons can be exchanged between the two hadrons, and showing that their effects cancel is the difficult step in the proofs of Collins, Soper and Sterman [1, 2]. The theorem holds for sufficiently inclusive observables, up to corrections suppressed by powers of the hard scale.

Drell–Yan at lowest order

The Drell–Yan process is the production of a lepton pair of large invariant mass M in a hadron collision, $pp \to l^+l^- + X$. At lowest order a quark from one hadron annihilates an antiquark from the other into a virtual photon.

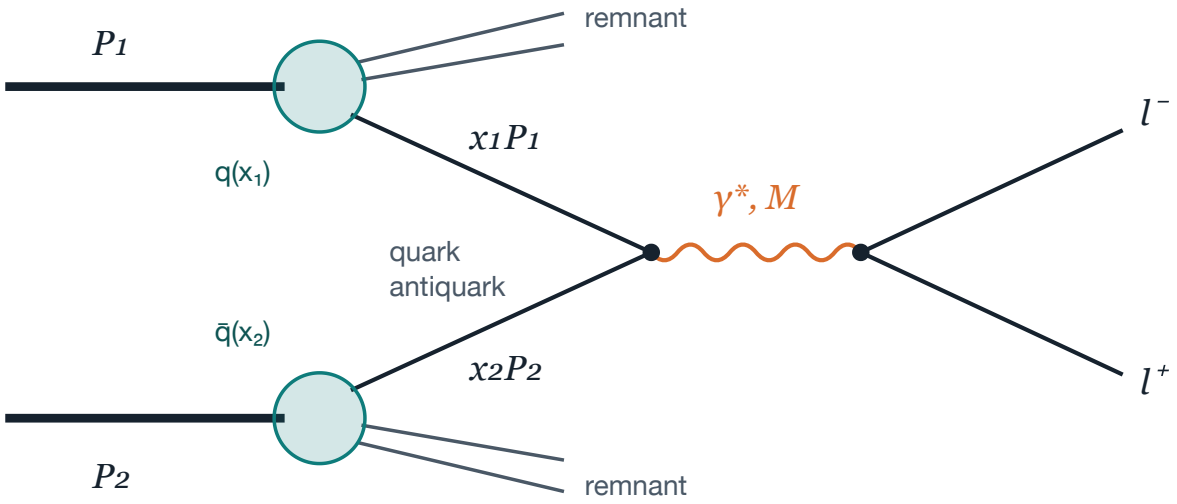

Figure 4. The Drell–Yan process at lowest order. Each proton supplies one parton with a probability given by its distribution, shown as a shaded blob. The quark and antiquark annihilate into a virtual photon of mass M, which decays to a lepton pair. The remnants of the protons do not affect the inclusive cross-section. The same picture with the photon replaced by a W or Z describes weak-boson production.

The parton cross-section. For $q(p_1) + \bar{q}(p_2) \to l^-(k_1) + l^+(k_2)$ with all masses neglected, the spin-averaged squared amplitude is $e^4 e_q^2 (1 + \cos^2\theta)$, where θ is the scattering angle in the centre-of-mass frame and e is the electron charge (derivation D7). Integrating over angles, and averaging over the colours of the incoming pair:

$$\hat{\sigma}(q\bar{q} \to l^+l^-) = \frac{4\pi\alpha^2}{3\hat{s}}\, e_q^2 \cdot \frac{1}{N_c}, \qquad N_c = 3. \tag{1.6.31}$$

Here α is the fine-structure constant and $\hat{s} = (p_1 + p_2)^2$. The colour factor is $1/3$ because the photon is colourless. Of the 3×3 colour combinations of a quark and an antiquark only the three in which the colours match can annihilate, and the average over initial colours divides by 9. The reverse process, $e^+e^- \to q\bar{q}$, has a factor N_c instead, from summing over final colours.

Kinematics. In the centre-of-mass frame of the two hadrons, with total energy $\sqrt{s}$, the partons have momenta $p_{1,2} = \frac{\sqrt{s}}{2}(x_{1,2}, 0, 0, \pm x_{1,2})$. The pair they produce has invariant mass and rapidity

$$M^2 = \hat{s} = x_1 x_2\, s, \qquad y = \frac{1}{2}\ln\frac{E + p_z}{E - p_z} = \frac{1}{2}\ln\frac{x_1}{x_2} \qquad \Longrightarrow \qquad x_{1,2} = \frac{M}{\sqrt{s}}\, e^{\pm y}. \tag{1.6.32}$$

Measuring the mass and rapidity of the lepton pair fixes both momentum fractions. There is no integral left: at lowest order the Drell–Yan process samples the distributions at two points.

The hadronic cross-section. The lepton pair has $\hat{s} = M^2$, so $d\hat{\sigma}/dM^2 = \hat{\sigma}\,\delta(\hat{s} - M^2)$. Insert this in Eq. (1.6.30) and change variables from (x_1, x_2) to (M^2, y), for which $dx_1\,dx_2 = dM^2\,dy/s$:

$$\frac{d^2\sigma}{dM^2\,dy} = \frac{4\pi\alpha^2}{9\,M^2\,s} \sum_q e_q^2 \Big[q(x_1, M^2)\,\bar{q}(x_2, M^2) + \bar{q}(x_1, M^2)\,q(x_2, M^2) \Big]. \tag{1.6.33}$$

The sum is over quark flavours. The two terms are the two ways of assigning the quark and the antiquark to the two hadrons. Integrating over rapidity gives the mass spectrum in terms of a *parton luminosity*:

$$\frac{d\sigma}{dM^2} = \frac{4\pi\alpha^2}{9\,M^4}\,\tau \sum_q e_q^2\,\mathcal{L}_{q\bar q}(\tau, M^2), \qquad \tau = \frac{M^2}{s},$$
$$\mathcal{L}_{q\bar q}(\tau,\mu^2) = \int_\tau^1 \frac{dx}{x}\Big[q(x,\mu^2)\,\bar q\Big(\frac{\tau}{x},\mu^2\Big) + \bar q(x,\mu^2)\,q\Big(\frac{\tau}{x},\mu^2\Big)\Big]. \tag{1.6.34}$$

The luminosity is the convolution of two distributions. It measures the flux of parton pairs with a given $\hat s = \tau s$, and it is the natural quantity for comparing what different PDF sets predict for the LHC. If the distributions did not depend on scale, $M^4\,d\sigma/dM^2$ would depend on τ alone. This is the Drell–Yan analogue of Bjorken scaling, and it is violated logarithmically for the same reason.

Drell–Yan at order α_S: the same poles

At order α_S there are virtual corrections to $q\bar q \to \gamma^*$, real emission $q\bar q \to \gamma^* g$, and a new channel, $qg \to \gamma^* q$. The calculation in d dimensions runs as in sections 3 and 4. Soft poles cancel between real and virtual terms. The collinear poles do not. Writing the parton-level cross-section as $e_q^2\big[\delta(1-z) + \frac{\alpha_S}{2\pi}\hat{\mathcal{F}}_1(z)\big]$ with $z = M^2/\hat s$, the result quoted from ESW, Eq. (9.20) [2], is

$$\hat{\mathcal{F}}_1^{q\bar q}(z) = 2\Big(-\frac{1}{\epsilon} - \ln 4\pi + \gamma_E + \ln\frac{M^2}{\mu^2}\Big)P_{qq}(z) + D_q(z),$$
$$\hat{\mathcal{F}}_1^{qg}(z) = \Big(-\frac{1}{\epsilon} - \ln 4\pi + \gamma_E + \ln\frac{M^2}{\mu^2}\Big)P_{qg}(z) + D_g(z). \tag{1.6.35}$$

Compare with Eq. (1.6.16). The pole multiplies the same P_{qq}, with the same accompanying constants. The factor 2 is there because either incoming parton can radiate. In the quark–gluon channel the pole multiplies P_{qg}: the gluon splits into a quark pair, one of which annihilates. Substituting the $\overline{\text{MS}}$ distributions of Eq. (1.6.18) for both hadrons removes every pole. No new subtraction is needed. This is universality verified by explicit calculation at one order, in one process.

What is left, D_q and D_g, is the Drell–Yan coefficient function. Its delta-function term in $\overline{\text{MS}}$ is $C_F\big(\frac{2\pi^2}{3} - 8\big)\delta(1-z)$ [2]. By the rule of section 6, with two incoming partons, the DIS-scheme coefficient is $D_q - 2C_q$, and its delta-function term is $C_F\big(1 + \frac{4\pi^2}{3}\big)\delta(1-z)$. With $\alpha_S = 0.2$ the first is a correction of -6% and the second of $+60\%$. The size of a correction, like the distributions themselves, is a statement about a scheme. Only the sum is physical.

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10. Which process measures what

This section goes through the main hadron-collider processes used in fits, says which combinations of distributions each one measures and why, and places them on the x–Q^2 plane.

W and Z production

Replace the photon by a weak boson. For $q\bar q' \to W$ the parton cross-section is $\hat\sigma = \frac{\pi}{3}\sqrt{2}\,G_F M_W^2\,|V_{qq'}|^2\,\delta(\hat s - M_W^2)$, with G_F the Fermi constant and $V_{qq'}$ the quark mixing matrix element [2]. The kinematics is Eq. (1.6.32) with $M = M_W$. The charge of the boson selects the flavours: W^+ is made mainly from $u\bar d$, and W^- from $d\bar u$. In a proton–proton collision the charge asymmetry at rapidity y is, neglecting the smaller contributions from strange and charm and with the same constant of proportionality for both charges,

$$A_W(y) = \frac{d\sigma_{W^+}/dy - d\sigma_{W^-}/dy}{d\sigma_{W^+}/dy + d\sigma_{W^-}/dy},$$
$$\frac{d\sigma_{W^+}}{dy} \propto u(x_1)\,\bar d(x_2) + \bar d(x_1)\,u(x_2), \qquad \frac{d\sigma_{W^-}}{dy} \propto d(x_1)\,\bar u(x_2) + \bar u(x_1)\,d(x_2), \tag{1.6.36}$$

At $y = 0$ both fractions equal $M_W/\sqrt{s}$, and if $\bar{u} = \bar{d}$ there, the asymmetry is $(u - d)/(u + d)$. It is positive because the proton has two valence up quarks and one down. Away from $y = 0$ one fraction is large and the other small, and the asymmetry measures the ratio u/d of valence quarks at large x together with $\bar{d}/\bar{u}$ at small x. Deep inelastic scattering on a proton with a photon measures the single combination $\sum e_q^2 (q + \bar{q})$ and cannot separate these. In practice the charged lepton from the decay is measured, not the W, and the lepton asymmetry is a known smearing of Eq. (1.6.36).

Z production adds $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$ with the weak couplings. Because strange quarks contribute to Z production directly and to W production only with charm, the ratio of W to Z cross-sections is sensitive to the strange sea. The inclusion of precise LHC W and Z data has changed the strange distribution in recent fits [3].

Jets and top pairs

At lowest order a jet cross-section is the scattering of two partons into two, $qq \to qq$, $qg \to qg$ and $gg \to gg$ among others, at order α_S^2. A jet of transverse momentum p_T at central rapidity needs $x_{1,2} \sim 2p_T/\sqrt{s}$. Unlike in DIS and Drell–Yan, the gluon enters at lowest order. Jet data therefore constrain the gluon directly, and mostly at large x [3].

Top-quark pairs are produced by $gg \to t\bar{t}$ and $q\bar{q} \to t\bar{t}$, with the gluon channel dominant at the LHC. The pair has mass at least $2m_t \approx 345$ GeV, which at 13 TeV and $y = 0$ means $x \geq 0.027$, and higher for heavier or more forward pairs. Top-pair data also constrain the gluon at large x, less strongly than jets [3].

Direct photons, from $qg \to \gamma q$, and the transverse-momentum spectrum of the Z, from $qg \to Zq$ and $q\bar{q} \to Zg$, give further handles on the gluon. W production with a charm quark, $gs \to Wc$, measures the strange distribution [3].

The table

The assignments below are those of the Snowmass review [3]. The values of x are either quoted from it or computed from Eq. (1.6.32) at $\sqrt{s} = 13$ TeV, as marked.

PROCESS	LOWEST-ORDER SUBPROCESS	MAINLY CONSTRAINS	WHERE IN x
Photon-exchange DIS, fixed target	$\gamma^* q \to q$	charge-weighted quarks; gluon through scaling violations	$x \gtrsim 0.007$ at our Q^2 cut, 280 GeV beam (computed)
DIS at HERA	$\gamma^* q \to q$, and W exchange	quarks and, through scaling violations, the gluon at small x	down to $x \sim 3 \times 10^{-5}$ at our Q^2 cut (computed)
Neutrino DIS, dimuon events	$W^+ s \to c$	strange	$x \gtrsim 0.01$ [3]
W, Z/γ^* at central rapidity	$q\bar{q} \to W, Z$	valence quarks; flavour structure of the sea	6×10^{-4} to 0.09 for M_Z, $\lvert y \rvert < 2.5$ (computed)
W, Z at forward rapidity (LHCb)	$q\bar{q} \to W, Z$	quarks and antiquarks at small and large x, strangeness	8×10^{-5} and 0.6 for M_Z, $y = 4.5$ (computed)
Inclusive jets, dijets	gg, qg, qq scattering	gluon at large x; minor constraints on the sea	$x \sim 2p_T/\sqrt{s}$
Z at high p_T, W/Z + jets	$qg \to Zq$, $q\bar{q} \to Zg$	gluon and quarks; valence and sea at higher x	higher x than inclusive W, Z [3]
Top pairs	$gg \to t\bar{t}$	gluon at large x, more weakly than jets	$x \geq 0.027$ at $y = 0$ (computed)
Direct photon	$qg \to \gamma q$	gluon at large x, more weakly than top and jets	
W + charm	$gs \to Wc$	strange	
Open charm and beauty at LHCb	$gg \to Q\bar{Q}$	gluon at small x	$x \sim 5 \times 10^{-6}$ [3]

The kinematic plane

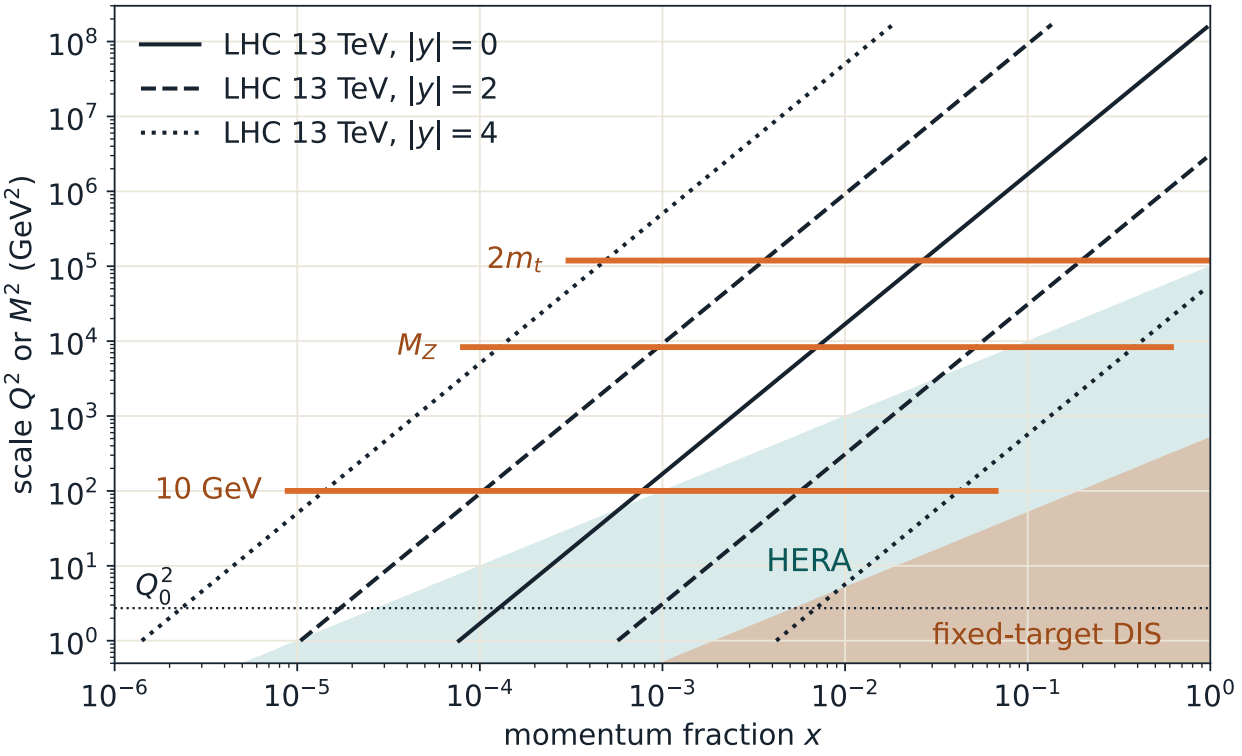

Figure 5. Where the data sit. Shaded: the regions allowed by kinematics for DIS, $Q^2 \leq x\,s$, at HERA ($\sqrt{s} = 318$ GeV) and for a fixed-target muon beam of 280 GeV. Black lines: Eq. (1.6.32) for the LHC at 13 TeV at fixed rapidity; each value of $|y| > 0$ gives two lines, one for each parton. Orange bars: the range of x covered for $|y| < 4.5$ at three masses of the produced system. The dotted line is the starting scale of our fits. DGLAP evolution connects any point to the points above it at equal or larger x. Computed from kinematics alone; no data are plotted.

Figure 5 shows why evolution is what makes a global fit possible. A Z boson at the LHC is produced at $Q^2 \approx 8 \times 10^3$ GeV^2 from partons with x between 10^{-4} and 0.6. Most of the DIS data sit orders of magnitude lower in Q^2. The two can be combined only because the distributions at M_Z are computed from those at Q_0, and because, by section 9, they are the same distributions.

Orders available

The status of the hard cross-sections, as stated in the Snowmass review of 2022 [3]:

- **DIS.** Partonic cross-sections are available at next-to-next-to-next-to-leading order (N³LO).
- **Hadron colliders, NNLO.** NNLO is available for all key processes. Predictions for inclusive W and Z, jets, W/Z + jets, top pairs and direct photons are made at NNLO in QCD with next-to-leading-order electroweak corrections.
- **Hadron colliders, N³LO.** Available for key inclusive cross-sections and some fully differential predictions. The corrections are of the order of a few per cent.
- **Not yet at NNLO.** W + charm needs extension to NNLO, and open charm and beauty at LHCb are described at next-to-leading order.

At the time of the review an N³LO cross-section was folded with NNLO distributions, since a consistent N³LO fit needs the four-loop splitting functions and N³LO cross-sections for the processes fitted. The review puts the resulting mismatch at several per cent for key LHC observables [3]. Chapter 3.1 takes this up.

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11. Where our fits use this

The fixed setup of our fits is theory 40000000: NNLO, FONLL-C, $Q_0 = 1.65$ GeV, $\alpha_S(M_Z) = 0.118$, $m_c = 1.51$ GeV, with data cuts $Q^2 > 3.49$ GeV^2 and $W^2 > 12.5$ GeV^2. Each item is a choice explained in this chapter.

- **NNLO, $\overline{\text{MS}}$.** The 52 parameters of the MSHT20 form describe $\overline{\text{MS}}$ distributions at Q_0. They have meaning only together with NNLO coefficient functions in the same scheme (section 6). The FK tables of chapter 2.1 contain those coefficient functions, folded with the evolution.
- **$\alpha_S(M_Z) = 0.118$.** The coupling at Q_0 follows by section 2. At two loops it is 0.327; the theory itself runs the coupling at NNLO, with one more term of Eq. (1.6.1).
- **FONLL-C for DIS.** Structure functions near the charm threshold, which the cut at 3.49 GeV^2 does not exclude, are computed with Eq. (1.6.28).

Section 10 is what our Phase B results measure. In the Fisher-information ladder of PPDF-32, a closure test, 12 of the 52 parameter directions are flat with the 19 DIS datasets alone: the data do not constrain them. DIS is blind to the flavour separation of the sea, as Phase A had concluded (PPDF-16). With the Drell–Yan, W and Z datasets added (54 datasets) the count falls to 9. With jets and direct photons (59) it falls to 8, and with the top-quark data (all 82) it stays at 8. On the real global data the fit has $\chi^2/N_{\text{dat}} = 1.2380$ for 4615 points (PPDF-35). One set of 52 numbers at Q_0 describing DIS, Drell–Yan, jets and top production together at that level is a direct test of Eq. (1.6.30).

Derivations

D1. The running coupling at one and two loops

Derives (1.6.2), (1.6.3), (1.6.4)

Write $t = \ln \mu^2$, so Eq. (1.6.1) is $d\alpha_S/dt = -b\,\alpha_S^2(1 + b'\alpha_S)$.

One loop. With $b' = 0$, $d\alpha_S/\alpha_S^2 = -b\,dt$. The left side is $-d(1/\alpha_S)$, so $1/\alpha_S(\mu^2) - 1/\alpha_S(\mu_0^2) = b\ln(\mu^2/\mu_0^2)$. Solving for $\alpha_S(\mu^2)$ gives Eq. (1.6.2). Choosing $\mu_0 = \Lambda$, the scale at which $1/\alpha_S = 0$, gives the first form of Eq. (1.6.4).

Two loops. Separate the variables and decompose into partial fractions:

$$\frac{d\alpha_S}{\alpha_S^2(1 + b'\alpha_S)} = \Big[\frac{1}{\alpha_S^2} - \frac{b'}{\alpha_S} + \frac{b'^2}{1 + b'\alpha_S}\Big]d\alpha_S = -b\,dt.$$

The three terms integrate to $-1/\alpha_S$, $-b'\ln\alpha_S$ and $b'\ln(1 + b'\alpha_S)$. Between μ_0 and μ:

$$-\frac{1}{\alpha_S(\mu^2)} + \frac{1}{\alpha_S(\mu_0^2)} + b'\ln\frac{1 + b'\alpha_S(\mu^2)}{\alpha_S(\mu^2)} - b'\ln\frac{1 + b'\alpha_S(\mu_0^2)}{\alpha_S(\mu_0^2)} = -b\ln\frac{\mu^2}{\mu_0^2}.$$

Changing the overall sign gives Eq. (1.6.3).

The expanded form. Let $\mu_0 \to \Lambda$ formally, collecting the terms that depend on μ_0 into the definition of Λ: $1/\alpha_S + b' \ln\left[b'\alpha_S/(1+b'\alpha_S)\right] = b\,L$ with $L = \ln(\mu^2/\Lambda^2)$. For large L the first approximation is $1/\alpha_S \simeq bL$. Inserting it in the logarithm, $b'\ln(b'\alpha_S) \simeq -b'\ln(bL/b')$, so $1/\alpha_S \simeq bL + b'\ln L + \text{constant}$. The constant can be absorbed by rescaling Λ, and the conventional choice is to remove it. Inverting, $\alpha_S = \frac{1}{bL}\left[1 - \frac{b'}{b}\frac{\ln L}{L}\right]$ up to terms of relative order $\ln^2 L/L^2$, which is the second form of Eq. (1.6.4). The check script solves it for Λ and finds that it reproduces the exact two-loop coupling to 0.1% at 20 GeV.

Numbers. For $n_f = 5$: $b = 23/12\pi = 0.610$ and $b' = 29/23\pi = 0.401$. For $n_f = 4$: $b = 25/12\pi = 0.663$ and $b' = 77/50\pi = 0.490$. At one loop, $1/\alpha_S(m_b) = 1/0.118 - 0.610\,\ln(91.2^2/4.92^2) = 8.475 - 3.563 = 4.912$, so $\alpha_S(m_b) = 0.204$. Then $1/\alpha_S(Q_0) = 4.912 - 0.663\,\ln(4.92^2/1.65^2) = 4.912 - 1.449 = 3.463$, so $\alpha_S(Q_0) = 0.289$. The two-loop values are the roots of Eq. (1.6.3), found numerically and confirmed by integrating Eq. (1.6.1) directly.

D2. Two-particle phase space in d dimensions

Derives (1.6.7), (1.6.8)

The phase space for the massless gluon r and quark l with total momentum $p+q$ is

$$\int d\Phi_2 = \int \frac{d^{d-1}r}{(2\pi)^{d-1}2E_r}\,\frac{d^{d-1}l}{(2\pi)^{d-1}2E_l}\,(2\pi)^d\,\delta^d(p+q-r-l).$$

The momentum integral. Work in the centre-of-mass frame, where $\vec{p}+\vec{q}=0$ and the total energy is $\sqrt{\hat{s}}$. The spatial delta function sets $\vec{l} = -\vec{r}$, so $E_l = E_r = |\vec{r}|$:

$$\int d\Phi_2 = \frac{1}{4(2\pi)^{d-2}} \int \frac{d^{d-1}r}{|\vec{r}|^2}\,\delta\big(\sqrt{\hat{s}} - 2|\vec{r}|\big).$$

Polar coordinates. In $d-1$ spatial dimensions, $d^{d-1}r = |\vec{r}|^{d-2}\,d|\vec{r}|\,d\Omega_{d-2}$. The energy delta function gives $\int d|\vec{r}|\,|\vec{r}|^{d-4}\,\delta(\sqrt{\hat{s}} - 2|\vec{r}|) = \frac{1}{2}(\hat{s}/4)^{-\epsilon}$, using $d-4 = -2\epsilon$.

The angles. Measure the polar angle θ from the direction of the incoming quark. The matrix element depends on no other angle, so the rest can be integrated: $d\Omega_{d-2} = (\sin\theta)^{d-3}\,d\theta\;\Omega_{d-3}$, where Ω_{d-3} is the area of the unit sphere in $d-2$ dimensions,

$$\Omega_{d-3} = \frac{2\pi^{(d-2)/2}}{\Gamma\left(\frac{d-2}{2}\right)} = \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}.$$

At $\epsilon = 0$ this is 2π, the circumference of a circle. With $v = (1-\cos\theta)/2$: $\sin^2\theta = 4v(1-v)$ and $(\sin\theta)^{d-3}d\theta = (\sin\theta)^{-2\epsilon}\,d\cos\theta = \big[4v(1-v)\big]^{-\epsilon}\,2\,dv$.

Collecting factors.

$$\int d\Phi_2 = \frac{1}{4(2\pi)^{2-2\epsilon}} \cdot \frac{1}{2}\left(\frac{\hat{s}}{4}\right)^{-\epsilon} \cdot \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \cdot 4^{-\epsilon} \cdot 2 \int_0^1 dv\,\big[v(1-v)\big]^{-\epsilon}.$$

The powers of 4 cancel. The numerical factor is $\frac{2\pi}{16\pi^2} = \frac{1}{8\pi}$, and the remaining powers of π are $(2\pi)^{2\epsilon}\pi^{-\epsilon} = (4\pi)^{\epsilon}$. This is Eq. (1.6.8).

The relation of v to invariants. In the same frame the incoming quark has energy $E_p = (\hat{s}+Q^2)/2\sqrt{\hat{s}}$, from $q^2 = -Q^2$. Then $2r\cdot p = 2\,\frac{\sqrt{\hat{s}}}{2}E_p(1-\cos\theta) = (\hat{s}+Q^2)\,v$. With $\hat{s} = 2\nu - Q^2$ and $2\nu = Q^2/z$, $\hat{s}+Q^2 = Q^2/z$ and $\hat{s} = Q^2(1-z)/z$, which are the relations of Eq. (1.6.7).

D3. The distributional expansion of $z^\epsilon(1-z)^{-1-\epsilon}$

Derives (1.6.13)

Let $f(z)$ be smooth on $0 \le z \le 1$ and take $\epsilon < 0$, so that all integrals converge. Add and subtract $f(1)$:

$$\int_0^1 dz\,\frac{z^\epsilon}{(1-z)^{1+\epsilon}}\,f(z) = f(1)\int_0^1 dz\,\frac{z^\epsilon}{(1-z)^{1+\epsilon}} + \int_0^1 dz\,\frac{z^\epsilon}{(1-z)^{\epsilon}}\,\frac{f(z)-f(1)}{1-z}.$$

The first term. The integral is an Euler Beta function, $B(1+\epsilon,-\epsilon) = \Gamma(1+\epsilon)\Gamma(-\epsilon)/\Gamma(1)$. Using $\Gamma(-\epsilon) = -\Gamma(1-\epsilon)/\epsilon$ and the reflection formula $\Gamma(1+\epsilon)\Gamma(1-\epsilon) = \pi\epsilon/\sin\pi\epsilon = 1 + \frac{\pi^2}{6}\epsilon^2 + \ldots$:

$$\int_0^1 dz\,\frac{z^\epsilon}{(1-z)^{1+\epsilon}} = -\frac{1}{\epsilon} - \frac{\pi^2}{6}\,\epsilon + O(\epsilon^3).$$

The pole is the soft divergence, regulated.

The second term. Because $f(z)-f(1)$ vanishes linearly at $z=1$, the integrand is finite there, and the prefactor can now be expanded: $z^\epsilon(1-z)^{-\epsilon} = 1 + \epsilon\ln z - \epsilon\ln(1-z) + O(\epsilon^2)$. Each term has at worst a logarithmic singularity at an end point, which is integrable. So the second term is

$$\int_0^1 dz\,\frac{f(z)-f(1)}{1-z} - \epsilon\int_0^1 dz\,\ln(1-z)\,\frac{f(z)-f(1)}{1-z} + \epsilon\int_0^1 dz\,\frac{\ln z}{1-z}\,\big[f(z)-f(1)\big] + O(\epsilon^2).$$

The first two integrals are, by definition, f integrated against $1/(1-z)_+$ and against $\big(\ln(1-z)/(1-z)\big)_+$. In the third, $\ln z/(1-z)$ is finite at $z=1$, so the two pieces can be separated: $\int_0^1 dz\,\frac{\ln z}{1-z} f(z) - f(1)\int_0^1 dz\,\frac{\ln z}{1-z}$, and $\int_0^1 dz\,\frac{\ln z}{1-z} = -\frac{\pi^2}{6}$.

Adding up. The terms proportional to $\epsilon f(1)$ are $-\frac{\pi^2}{6}\epsilon f(1)$ from the Beta function and $+\frac{\pi^2}{6}\epsilon f(1)$ from the last integral. They cancel. What remains is

$$\int_0^1 dz\,\frac{z^\epsilon f(z)}{(1-z)^{1+\epsilon}} = -\frac{f(1)}{\epsilon} + \int_0^1 dz\,\frac{f(z)}{(1-z)_+} - \epsilon\int_0^1 dz\,\Big[\Big(\frac{\ln(1-z)}{1-z}\Big)_+ - \frac{\ln z}{1-z}\Big]f(z) + O(\epsilon^2),$$

which holds for every smooth f and is therefore Eq. (1.6.13). The check script evaluates both sides for a test function at $\epsilon = -0.02$ and -0.01; the difference falls by a factor of 4.06, as an $O(\epsilon^2)$ remainder should.

D4. The angular integral and the Laurent expansion

Derives (1.6.11), (1.6.12), (1.6.14)

Beta functions. The four integrals needed in Eq. (1.6.10) are of the form $\int_0^1 dv\,v^{a-1}(1-v)^{c-1} = \Gamma(a)\Gamma(c)/\Gamma(a+c)$. Reduce each with $\Gamma(x+1) = x\Gamma(x)$ to a multiple of $K \equiv \Gamma(1-\epsilon)^2/\Gamma(1-2\epsilon)$:

$$\int_0^1 dv\,[v(1-v)]^{-\epsilon}\,v = \frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-2\epsilon)} = \frac{K}{2(1-2\epsilon)},$$
$$\int_0^1 dv\,[v(1-v)]^{-\epsilon}\,\frac{1}{v} = \frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} = -\frac{K}{\epsilon},$$
$$\int_0^1 dv\,[v(1-v)]^{-\epsilon}\,\frac{1-v}{v} = \frac{\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)} = -\frac{(1-\epsilon)\,K}{\epsilon(1-2\epsilon)},$$
$$\int_0^1 dv\,[v(1-v)]^{-\epsilon} = \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)} = \frac{K}{1-2\epsilon}.$$

The second and third require $\epsilon < 0$ and are continued analytically from there. The poles come from $\Gamma(-\epsilon)$, that is, from $v \to 0$.

Assembling. Insert these into the bracket of Eq. (1.6.10) and take out a factor $K/(1-z)$:

$$\int_0^1 dv\,[\dots] = \frac{K}{1-z}\left[\frac{1-\epsilon}{2(1-2\epsilon)} - \frac{(1-\epsilon)(1-z)^2}{\epsilon} - \frac{2z(1-\epsilon)}{\epsilon(1-2\epsilon)} + \frac{2\epsilon(1-z)}{1-2\epsilon}\right] = \frac{K}{1-z}\,N(z,\epsilon).$$

The prefactor $1/\Gamma(1-\epsilon)$ of Eq. (1.6.10) times K gives $\Gamma(1-\epsilon)/\Gamma(1-2\epsilon)$, and $(1-z)^{-\epsilon}/(1-z) = (1-z)^{-1-\epsilon}$. This is Eq. (1.6.11).

Expanding N. Use $\frac{1-\epsilon}{1-2\epsilon} = 1 + \epsilon + 2\epsilon^2 + \dots$:

$$N = -\frac{(1-z)^2}{\epsilon} + (1-z)^2 + \frac{1}{2} + \frac{\epsilon}{2} - \frac{2z}{\epsilon} - 2z - 4z\,\epsilon + 2\epsilon(1-z) + O(\epsilon^2).$$

The pole terms combine as $-[(1-z)^2 + 2z]/\epsilon = -(1+z^2)/\epsilon$. The finite terms are $(1-z)^2 + \frac{1}{2} - 2z = z^2 - 4z + \frac{3}{2}$. The terms linear in ϵ are $\frac{1}{2} - 4z + 2 - 2z = \frac{5}{2} - 6z$. This is Eq. (1.6.12).

The product. Write Eq. (1.6.13) as $-\delta(1-z)/\epsilon + D_0 + \epsilon D_1$ and Eq. (1.6.12) as $N_{-1}/\epsilon + N_0 + \epsilon N_1$. A delta function sets $z = 1$ in whatever multiplies it, where $N_{-1} = -2$, $N_0 = -\frac{3}{2}$, $N_1 = -\frac{7}{2}$. The terms that survive as $\epsilon \to 0$ are

$$-\frac{\delta(1-z)}{\epsilon}\Big[\frac{-2}{\epsilon} - \frac{3}{2} - \frac{7}{2}\,\epsilon\Big] + \frac{D_0\,N_{-1}}{\epsilon} + D_0 N_0 + D_1 N_{-1}.$$

The first group is $\big(\frac{2}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{7}{2}\big)\delta(1-z)$. The second is $-\frac{1}{\epsilon}\frac{1+z^2}{(1-z)_+}$. For the third, $z^2 - 4z + \frac{3}{2} = (1-z)(3-z) - \frac{3}{2}$, so $D_0 N_0 = 3 - z - \frac{3}{2}\frac{1}{(1-z)_+}$. The fourth is $(1+z^2)\big(\frac{\ln(1-z)}{1-z}\big)_+ - \frac{1+z^2}{1-z}\ln z$. Together these are Eq. (1.6.14). Throughout, a smooth function $w(z)$ multiplying a plus distribution acts on the test function first: $\int_0^1 dz\,w\,[h]_+ f = \int_0^1 dz\,h\,[w(z)f(z) - w(1)f(1)]$. This is the sense in which $\frac{1+z^2}{(1-z)_+}$ appears in P_{qq}.

Check. The script integrates the exact $z^\epsilon(1-z)^{-1-\epsilon}N(z,\epsilon)$ against a test function at $\epsilon = -0.004$ and -0.002 and subtracts the right side of Eq. (1.6.14). The remainder halves, as an $O(\epsilon)$ term should.

D5. Changing scheme

Derives (1.6.24)

Work with moments, where convolutions are products, and write $a = \alpha_S/2\pi$, so that Eq. (1.6.1) at one loop is $da/d\ln\mu^2 = -2\pi b\,a^2$.

Evolution. A non-singlet $\overline{\text{MS}}$ moment obeys $dq/d\ln\mu^2 = a\big(\gamma^{(0)} + a\,\gamma^{(1)}\big)q$, by Eqs. (1.4.25) and (1.4.34). The DIS-scheme moment is $q^{\text{DIS}} = (1 + a\,C_q)\,q$, with $C_q = C_q(N)$ independent of scale. Differentiate:

$$\frac{dq^{\rm DIS}}{d\ln\mu^2} = \frac{da}{d\ln\mu^2}\,C_q\,q + (1+a\,C_q)\,a\big(\gamma^{(0)} + a\,\gamma^{(1)}\big)q = a\Big[\gamma^{(0)} + a\big(\gamma^{(1)} - 2\pi b\,C_q\big)\Big]q^{\rm DIS} + O(a^3).$$

In the last step $q = q^{\rm DIS} + O(a)$ was used in the term that is already of order a^2. This is Eq. (1.6.24). The shift comes entirely from the running of the coupling in the factor $1 + aC_q$. In the singlet sector the distributions are a two-component vector and C a matrix, and there is an additional term, the commutator of C with the matrix of $\gamma^{(0)}$.

Momentum. The second moments of Eq. (1.6.23) give $\Sigma^{\rm DIS}(2) + g^{\rm DIS}(2) = \Sigma(2) + g(2)$ identically, since the bracket added to the quarks, summed over flavours, is the bracket subtracted from the gluon.

An observable. Eq. (1.6.22) for F_L is already of order a. Replacing the distributions by their DIS-scheme counterparts changes each by a term of order a, and so changes F_L at order a^2. No change of coefficient function is needed at this order, which is the sense in which the leading coefficient functions for F_L are scheme independent.

Two incoming partons. For the Drell–Yan quark–antiquark channel, $\sigma \propto q\,\bar q\,(1 + a\,D_q)$ in moment space. With $q = q^{\rm DIS}(1 - aC_q)$ and the same for $\bar q$, $\sigma \propto q^{\rm DIS}\bar q^{\rm DIS}\big(1 + a\,[D_q - 2C_q]\big)$. For the delta-function terms: $\big(\frac{2\pi^2}{3} - 8\big) + 2\big(\frac{\pi^2}{3} + \frac{9}{2}\big) = \frac{4\pi^2}{3} + 1$.

D6. The massless limit of the massive coefficient function

Derives (1.6.27), (1.6.29)

The logarithm. For $\lambda = m_c^2/Q^2 \to 0$ at fixed $z < 1$, $\beta_c^2 = 1 - \frac{4\lambda z}{1-z}$ gives $1 - \beta_c \simeq \frac{2\lambda z}{1-z}$ and $1 + \beta_c \simeq 2$. Hence

$$\ln\frac{1+\beta_c}{1-\beta_c} \;\to\; \ln\frac{1-z}{\lambda z} = \ln\frac{Q^2}{m_c^2} + \ln\frac{1-z}{z}.$$

The limit. Set $\lambda = 0$ and $\beta_c = 1$ everywhere else in $C_g^{\,m}$:

$$C_g^{\,m} \;\to\; T_R\Big[\big(z^2 + (1-z)^2\big)\Big(\ln\frac{Q^2}{m_c^2} + \ln\frac{1-z}{z}\Big) + 8z(1-z) - 1\Big] = P_{qg}(z)\ln\frac{Q^2}{m_c^2} + C_g(z),$$

with $P_{qg} = T_R[z^2 + (1-z)^2]$ and C_g as in Eq. (1.6.21). This is Eq. (1.6.27). The logarithm is the collinear one. In $\gamma^* g \to c\bar c$ the exchanged quark has virtuality of order m_c^2 at least, so the integral $\int d|k^2|/|k^2|$ of chapter 1.4 runs from m_c^2 to Q^2.

The threshold. The pair has $\hat s = Q^2(1-z)/z$ by Eq. (1.6.7), and needs $\hat s \geq 4m_c^2$. This is $z \leq 1/(1+4\lambda)$, where $\beta_c = 0$ and $C_g^{\,m}$ vanishes. In the massless limit the upper limit becomes 1.

The perturbative charm distribution. With $c(x, m_c^2) = 0$ and only the gluon term of Eq. (1.4.23) as a source, $\partial c/\partial\ln\mu^2 = \frac{\alpha_S}{2\pi}P_{qg}\otimes g$. Over a range of μ in which α_S and g change by terms of relative order α_S, the right side is constant to leading order and the solution is linear in $\ln\mu^2$: Eq. (1.6.29).

D7. The Drell–Yan cross-section at lowest order

Derives (1.6.31), (1.6.33), (1.6.34)

The amplitude. For $q(p_1)\,\bar q(p_2) \to \gamma^* \to l^-(k_1)\,l^+(k_2)$, with quark charge $e_q e$ and photon propagator $-ig_{\alpha\beta}/\hat s$:

$$\mathcal{M} = \frac{e_q\,e^2}{\hat s}\,\big[\bar v(p_2)\gamma^\alpha u(p_1)\big]\,\big[\bar u(k_1)\gamma_\alpha v(k_2)\big].$$

Squaring. Sum over final spins and average over the four initial spin states. With massless spinors each sum gives a trace:

$$\overline{\sum}|\mathcal{M}|^2 = \frac{e_q^2 e^4}{4\hat{s}^2}\,\mathrm{Tr}\left[\not{p}_2\gamma^\alpha \not{p}_1\gamma^\beta\right]\mathrm{Tr}\left[\not{k}_1\gamma_\alpha \not{k}_2\gamma_\beta\right].$$

Each trace is $4\left[a^\alpha b^\beta + a^\beta b^\alpha - g^{\alpha\beta}a\cdot b\right]$. Contracting the two, the terms with $g^{\alpha\beta}$ cancel among themselves in four dimensions and

$$\overline{\sum}|\mathcal{M}|^2 = \frac{8\,e_q^2 e^4}{\hat{s}^2}\Big[(p_1\cdot k_1)(p_2\cdot k_2) + (p_1\cdot k_2)(p_2\cdot k_1)\Big].$$

Angles. In the centre-of-mass frame every particle has energy $\sqrt{\hat{s}}/2$. With θ the angle between $\vec{k}_1$ and $\vec{p}_1$: $p_1\cdot k_1 = p_2\cdot k_2 = \frac{\hat{s}}{4}(1-\cos\theta)$ and $p_1\cdot k_2 = p_2\cdot k_1 = \frac{\hat{s}}{4}(1+\cos\theta)$. So

$$\overline{\sum}|\mathcal{M}|^2 = \frac{8e_q^2e^4}{\hat{s}^2}\cdot\frac{\hat{s}^2}{16}\left[(1-\cos\theta)^2 + (1+\cos\theta)^2\right] = e_q^2e^4\,(1+\cos^2\theta).$$

The cross-section. For massless two-to-two scattering, $d\hat{\sigma}/d\Omega = \overline{\sum}|\mathcal{M}|^2/64\pi^2\hat{s}$. With $\int d\Omega\,(1+\cos^2\theta) = 2\pi\left(2+\frac{2}{3}\right) = \frac{16\pi}{3}$ and $e^4 = 16\pi^2\alpha^2$:

$$\hat{\sigma} = \frac{16\pi^2\alpha^2 e_q^2}{64\pi^2\hat{s}}\cdot\frac{16\pi}{3} = \frac{4\pi\alpha^2}{3\hat{s}}\,e_q^2.$$

Colour. The quark has colour index i and the antiquark j, each running over three values. The photon vertex is diagonal in colour, δ_{ij}, so the squared amplitude summed over colours carries $\sum_{ij}\delta_{ij}\delta_{ij} = 3$. Averaging over the 9 initial colour states gives $3/9 = 1/N_c$. This is Eq. (1.6.31).

From partons to hadrons. With $d\hat{\sigma}/dM^2 = \hat{\sigma}\,\delta(x_1x_2s - M^2)$, Eq. (1.6.30) gives

$$\frac{d\sigma}{dM^2} = \frac{4\pi\alpha^2}{9M^2}\sum_q e_q^2\int dx_1\,dx_2\;\delta(x_1x_2s - M^2)\left[q(x_1)\bar{q}(x_2) + \bar{q}(x_1)q(x_2)\right].$$

Change variables to $\hat{s} = x_1x_2s$ and $y = \frac{1}{2}\ln(x_1/x_2)$. From $x_{1,2} = \sqrt{\hat{s}/s}\,e^{\pm y}$, the Jacobian is

$$\frac{\partial(x_1,x_2)}{\partial(\hat{s},y)} = \begin{vmatrix} \frac{x_1}{2\hat{s}} & x_1 \\ \frac{x_2}{2\hat{s}} & -x_2 \end{vmatrix} = -\frac{x_1x_2}{\hat{s}} = -\frac{1}{s},$$

so $dx_1\,dx_2 = d\hat{s}\,dy/s$. The delta function does the $\hat{s}$ integral and leaves the integrand of the y integral, which is Eq. (1.6.33).

The luminosity. To integrate Eq. (1.6.33) over y at fixed $\tau = M^2/s$, use $x_1 = \sqrt{\tau}e^y$ as the variable: $dy = dx_1/x_1$ and $x_2 = \tau/x_1$, with x_1 running from τ to 1 so that $x_2 \le 1$. Then $\int dy\,q(x_1)\bar{q}(x_2) = \int_\tau^1 \frac{dx}{x}\,q(x)\,\bar{q}(\tau/x)$, and with $1/(M^2s) = \tau/M^4$ this is Eq. (1.6.34).

References

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