1.4 Scaling violations: gluon emission, factorisation and DGLAP

Why structure functions change slowly with Q², derived from one gluon emission: the collinear logarithm, its absorption into the distributions, and the evolution equations that follow.

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1. The question

The parton model of chapters 1.2 and 1.3 says that a virtual photon of large Q^2 scatters off a single free quark carrying a fraction x of the proton's momentum. Its central prediction is that the structure function depends on x alone:

$$F_2(x, Q^2) \;\longrightarrow\; F_2(x) = x \sum_{q,\bar{q}} e_q^2 \, q(x). \tag{1.4.1}$$

Here e_q is the quark's electric charge in units of e, and $q(x)\,dx$ is the number of quarks of that flavour with momentum fraction between x and $x + dx$. This is Bjorken scaling.

The data do not quite scale. At fixed x above about 0.2, F_2 falls slowly as Q^2 grows. At small x it rises. The change is slow, logarithmic in Q^2, and it is the same in every experiment.

This chapter explains that pattern from first principles. The argument has four steps, and each one is derived in full, either in the text or in the numbered derivations at the end:

1. A quark can radiate a gluon before it is struck. The probability carries a factor $\alpha_S \, dk_T^2/k_T^2$, where k_T is the transverse momentum the quark picks up. Integrated up to the kinematic limit $k_T^2 \sim Q^2$, this gives a term $\alpha_S \ln Q^2$. That logarithm is what breaks scaling.
2. The lower end of the same integral diverges. The divergence comes from gluons emitted parallel to the quark, which is long-distance physics that perturbation theory cannot describe.
3. The divergence is absorbed into the definition of the quark distribution, which thereby acquires a dependence on a scale μ. This step is factorisation.
4. The physical structure function cannot depend on the arbitrary scale μ. That requirement fixes how $q(x, \mu^2)$ changes with μ. The result is the DGLAP equation.

So the distributions themselves cannot be computed, but their change with scale can. That is why one fit at one scale predicts every hard process at every other scale, and it is what our own fits rely on (section 10).

The chapter follows the line of Ellis, Stirling and Webber, chapter 4.3 [1], with the dimensional-regularisation version in Moore's thesis, section 1.1 [2], as a cross-check. Every formula stated here has been checked numerically; the script is linked at the end.

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2. The set-up: a quark as the target

Light-cone vectors

Take the proton to move fast along the $+z$ axis and neglect its mass. Introduce two light-like vectors, p along the proton's momentum and n pointing the opposite way:

$$p^\mu = (P, 0, 0, P), \qquad n^\mu = \Big(\frac{1}{2P}, 0, 0, -\frac{1}{2P}\Big), \qquad p^2 = n^2 = 0, \quad n\cdot p = 1. \tag{1.4.2}$$

Any four-vector is a combination of p, n and a two-dimensional transverse vector. The photon momentum is

$$q^\mu = \nu\, n^\mu + q_T^\mu, \qquad \nu = p\cdot q, \qquad q_T^2 = Q^2, \qquad x = \frac{Q^2}{2\nu}. \tag{1.4.3}$$

Throughout, k_T^2 and q_T^2 denote the positive squared lengths of the transverse vectors.

The hadronic tensor $W^{\alpha\beta}$ of chapter 1.1 contains two structure functions. Contracting it with $n_\alpha n_\beta$ picks out F_2:

$$F_2 = \nu\, n_\alpha n_\beta\, W^{\alpha\beta}. \tag{1.4.4}$$

The quark structure function

To find what QCD does to the parton model, replace the proton by a single quark of momentum p and compute *its* structure function, written $\hat{F}_2$. The hat marks a quantity for a quark target. The proton's F_2 is then $\hat{F}_2$ folded with the distribution of quarks in the proton.

At lowest order the process is $\gamma^*(q) + q(p) \to q(l)$. The amplitude is $-ie_q\,\bar{u}(l)\gamma^\alpha u(p)$, and averaging over the incoming spin gives $n_\alpha n_\beta \overline{\sum}|\mathcal{M}|^2_{\alpha\beta} = 4e_q^2$. The one-particle phase space is $2\pi\,\delta((p+q)^2)$, and the hadronic tensor carries a conventional factor $1/4\pi$. Using $(p+q)^2 = 2\nu - Q^2$:

$$\hat{F}_2(x) = \nu\cdot\frac{1}{4\pi}\cdot 2\pi\,\delta(2\nu - Q^2)\cdot 4e_q^2 = e_q^2\,\delta(1-x). \tag{1.4.5}$$

A free quark of momentum p can only absorb the photon if $x = 1$. Folded with $q(\xi)$ for a quark of momentum ξp, this reproduces the parton model, Eq. (1.4.1).

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3. One gluon: where the logarithm comes from

Now let the quark emit a gluon of momentum r before it absorbs the photon:

$$\gamma^*(q) + q(p) \;\to\; g(r) + q(l).$$

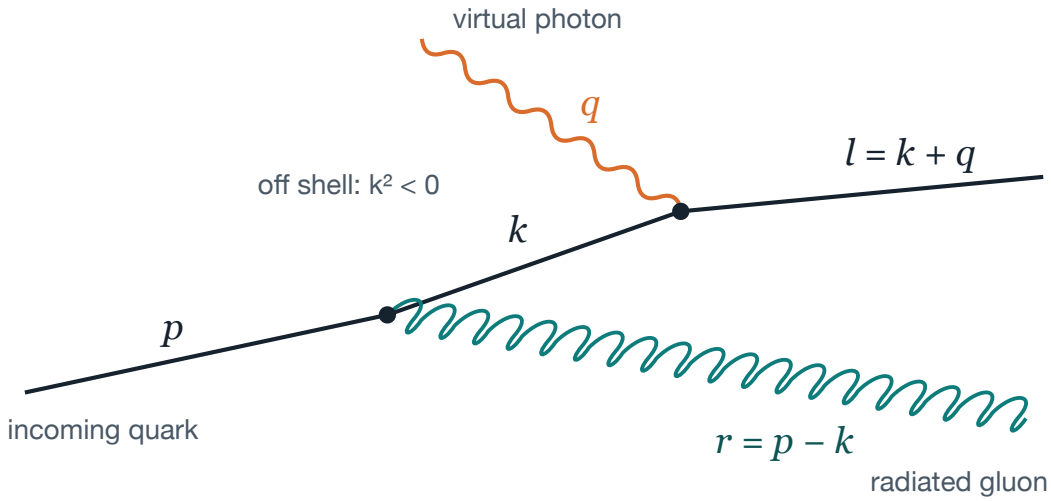

Figure 1. The emission that breaks scaling. The quark radiates a gluon, which leaves it off its mass shell with $k^2 < 0$ and with transverse momentum k_T, and only then meets the photon. The three other squared diagrams, in which the gluon is attached to the outgoing quark, give no logarithm.

Kinematics

Write the momentum of the struck quark line as

$$k^\mu = \xi\, p^\mu + \frac{k_T^2 - |k^2|}{2\xi}\, n^\mu + k_T^\mu. \tag{1.4.6}$$

The quark keeps a fraction ξ of the original momentum, it has transverse momentum k_T, and it is off shell by $k^2 = -|k^2|$. The gluon and the outgoing quark are real particles, so $r^2 = (p-k)^2 = 0$ and $l^2 = (k+q)^2 = 0$. These two conditions fix two of the variables:

$$k_T^2 = (1-\xi)\,|k^2|, \qquad \xi = x + \frac{|k^2| + 2\,q_T\cdot k_T}{2\nu}. \tag{1.4.7}$$

The first relation says that transverse momentum and virtuality go together: the quark is far off shell exactly when it has recoiled hard against the gluon. The second says that $\xi \to x$ when the virtuality is small compared with ν.

The matrix element

The amplitude for the diagram of Figure 1 is

$$\mathcal{M}^{\alpha} = -ig\, e_q\, \bar{u}(l)\, \gamma^{\alpha}\, \frac{\not{k}}{k^2}\, \not{\varepsilon} t^{A}\, u(p),$$

with g the strong coupling, ε the gluon polarisation and t^A the colour matrix. Squaring it gives two propagators, a factor $1/k^4$. If that survived, the integral over $|k^2|$ would be dominated by small virtuality and cut off there, and there would be no logarithm. It does not survive. Summing over the gluon's two physical polarisations and projecting onto F_2 (derivation D2), the trace supplies one power of $|k^2|$ in the numerator:

$$\frac{1}{4\pi}\, n_{\alpha} n_{\beta} \overline{\sum} |\mathcal{M}|^2_{\alpha\beta} = \frac{8\, e_q^2\, \alpha_S}{|k^2|}\, \xi\, \hat{P}(\xi), \qquad \hat{P}(\xi) = C_F\, \frac{1+\xi^2}{1-\xi}. \tag{1.4.8}$$

Here $\alpha_S = g^2/4\pi$ and $C_F = 4/3$ is the colour factor for a quark emitting a gluon. The function $\hat{P}(\xi)$ is the unregularised quark splitting function. Its form is specific to the quark–gluon vertex of QCD: the $1+\xi^2$ comes from a spin-½ quark coupled to a spin-1 gluon.

The integral

Putting the matrix element into the two-particle phase space (derivation D1) and doing the angular integral gives

$$\hat{F}_2 = e_q^2\, \frac{\alpha_S}{2\pi^2} \int_0^{2\nu} \frac{d|k^2|}{|k^2|} \int_{\xi_-}^{\xi_+} d\xi\; \frac{\xi\, \hat{P}(\xi)}{\sqrt{(\xi_+ - \xi)(\xi - \xi_-)}}, \tag{1.4.9}$$

$$\xi_{\pm} = x + z - 2zx \pm \sqrt{4x(1-x)\, z(1-z)}, \qquad z = \frac{|k^2|}{2\nu}.$$

Two features of Eq. (1.4.9) carry the physics.

- **The virtuality is not confined to small values.** The integral runs all the way to $|k^2| = 2\nu$, the kinematic limit. Nothing in the theory provides a smaller scale to cut it off.
- **The measure is $d|k^2|/|k^2|$.** Each decade of virtuality contributes equally. The integral diverges logarithmically at the lower end.

To extract the logarithm, cut the integral off at a small virtuality κ^2. For $z \to 0$ the limits close in, $\xi_\pm \to x$, and $\int d\xi / \sqrt{(\xi_+ - \xi)(\xi - \xi_-)} = \pi$, so

$$\hat{F}_2\Big|_{\log} = e_q^2 \, \frac{\alpha_S}{2\pi} \, x \, \hat{P}(x) \int_{\kappa^2}^{2\nu} \frac{d|k^2|}{|k^2|} = e_q^2 \, \frac{\alpha_S}{2\pi} \, x \, \hat{P}(x) \, \ln \frac{2\nu}{\kappa^2}. \tag{1.4.10}$$

In the light-cone gauge used for the polarisation sum, this diagram is the only one with a logarithm. The others have at most one propagator that can become singular, and a compensating factor of k_T from the vertex, so they give finite corrections. Writing $\ln 2\nu = \ln Q^2 - \ln x$ and collecting everything that is not a logarithm of Q^2 into a function $C(x)$:

$$\hat{F}_2(x, Q^2) = e_q^2 \, x \left[\delta(1-x) + \frac{\alpha_S}{2\pi} \Big(\hat{P}(x) \, \ln \frac{Q^2}{\kappa^2} + C(x) \Big) \right]. \tag{1.4.11}$$

Beyond leading order the structure function of a quark depends on Q^2. Bjorken scaling is broken by a logarithm, with a coefficient that QCD fixes.

> **Why a logarithm.** In a renormalisable gauge theory the probability to radiate at transverse momentum k_T falls only as dk_T^2/k_T^2. In a theory where it fell as $dk_T^2 \, m^2/k_T^4$, emission would stop at $k_T \sim m$ and structure functions would scale exactly once $Q^2 \gg m^2$. Logarithmic scaling violation is therefore direct evidence for a point-like vector coupling between quarks and gluons.

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4. Virtual gluons and the plus prescription

Equation (1.4.11) is incomplete in one respect. $\hat{P}(x)$ diverges as $x \to 1$, where the emitted gluon is soft. Real emission alone would give an infinite result when integrated over x.

The missing piece is the set of virtual diagrams at the same order in α_S: the gluon loop on each quark leg and the correction to the photon vertex. In all of them the final state is a single quark, so they carry the same $\delta((p+q)^2)$ as the lowest-order term and contribute only at $x = 1$:

$$\hat{P}(x) \;\longrightarrow\; \hat{P}(x) + K \, \delta(1-x).$$

The constant K can be found without computing a loop. Emitting a gluon does not change the number of quarks. The integral of the quark distribution over x therefore cannot depend on Q^2, which requires the full splitting function to integrate to zero. This fixes K and, with it, the meaning of the singularity at $x = 1$ (derivation D3):

$$P_{qq}(x) = C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\,\delta(1-x) \right]. \tag{1.4.12}$$

The plus distribution is defined by its integral against any smooth function f:

$$\int_0^1 dx\, \frac{f(x)}{(1-x)_+} = \int_0^1 dx\, \frac{f(x)-f(1)}{1-x}, \qquad \frac{1}{(1-x)_+} = \frac{1}{1-x} \ \text{ for } x < 1. \tag{1.4.13}$$

The subtraction of $f(1)$ is the virtual contribution. Soft real gluons add quarks just below $x = 1$ at the same rate at which virtual gluons remove them at $x = 1$, and the two infinities cancel. In dimensional regularisation the same cancellation is explicit: the real and virtual terms each have a double pole $2/\epsilon^2$ with opposite signs (Moore, Eqs. 1.69 and 1.70 [2]).

The soft divergence cancels. The collinear logarithm of Eq. (1.4.10) does not.

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5. Factorisation: absorbing the collinear divergence

What the divergence means

The logarithm in Eq. (1.4.10) diverges as the cut-off $\kappa^2 \to 0$. The divergent region is $k_T \to 0$: a gluon emitted parallel to the quark. A divergence of this kind does not cancel, because the photon can tell a quark from a collinear quark–gluon pair with the same total momentum. The pair's quark carries less momentum, so it scatters at a different x.

The limit $k_T \to 0$ is long-distance physics. A quark with tiny virtuality lives for a long time before it is struck, and over that time the coupling is strong and perturbation theory fails. The cut-off κ marks the place where the calculation stops being trustworthy. Nothing physical should depend on it.

The bare and the renormalised distribution

To get the proton's structure function, fold the quark's $\hat{F}_2$ with a distribution q_0 of quarks in the proton, as in the parton model:

$$\begin{aligned} F_2(x,Q^2) = x \sum_{q,\bar{q}} e_q^2 \Bigg[& q_0(x) \\ & + \frac{\alpha_S}{2\pi} \int_x^1 \frac{d\xi}{\xi}\, q_0(\xi) \left\{ P_{qq}\!\left(\frac{x}{\xi}\right) \ln \frac{Q^2}{\kappa^2} + C\!\left(\frac{x}{\xi}\right) \right\} + \dots \Bigg]. \end{aligned} \tag{1.4.14}$$

A quark of momentum fraction ξ radiates and ends with fraction x, so the splitting function is evaluated at the ratio x/ξ. The measure $d\xi/\xi$ comes with it.

Now treat q_0 exactly as the bare coupling is treated in renormalisation. It is not observable. Define a renormalised distribution at a *factorisation scale* μ that absorbs the dependence on the cut-off:

$$q(x,\mu^2) = q_0(x) + \frac{\alpha_S}{2\pi}\int_x^1 \frac{d\xi}{\xi}\, q_0(\xi)\left\{P_{qq}\Big(\frac{x}{\xi}\Big)\ln\frac{\mu^2}{\kappa^2} + C\Big(\frac{x}{\xi}\Big)\right\} + \dots \tag{1.4.15}$$

Splitting the logarithm as $\ln(Q^2/\kappa^2) = \ln(Q^2/\mu^2) + \ln(\mu^2/\kappa^2)$ and working to first order in α_S:

$$F_2(x,Q^2) = x\sum_{q,\bar q} e_q^2 \int_x^1 \frac{d\xi}{\xi}\, q(\xi,\mu^2)\left\{\delta\Big(1-\frac{x}{\xi}\Big) + \frac{\alpha_S}{2\pi}P_{qq}\Big(\frac{x}{\xi}\Big)\ln\frac{Q^2}{\mu^2} + \dots\right\}. \tag{1.4.16}$$

The cut-off has gone. The structure function is a convolution of two factors:

- $q(\xi,\mu^2)$ contains all the physics at transverse momenta below μ. It cannot be calculated, and is measured.
- The bracket contains the physics between μ and Q. It is calculable in perturbation theory and is free of large logarithms when $\mu \sim Q$.

This separation is the factorisation theorem. It has been shown here to first order in α_S; proofs exist to all orders [1, 2]. The same distributions then enter every hard process, which is what makes them worth measuring (chapter 1.6).

Schemes

Factorisation fixes how the logarithm is absorbed. It leaves open how much of the finite part C goes with it. That choice is the *factorisation scheme*.

- **DIS scheme.** Absorb all of C, as in Eq. (1.4.15). Then $F_2 = x\sum e_q^2\, q(x,Q^2)$ holds to all orders, by construction.
- $\overline{\text{MS}}$ **scheme.** Regulate dimensionally, so that the logarithm of the cut-off becomes a pole $-1/\epsilon$, and absorb only the pole together with the constants $\ln 4\pi - \gamma_E$ that always accompany it. A finite *coefficient function* is left behind:

$$F_2(x,Q^2) = x\sum_{q,\bar q} e_q^2 \int_x^1 \frac{d\xi}{\xi}\, q(\xi,Q^2)\left\{\delta\Big(1-\frac{x}{\xi}\Big) + \frac{\alpha_S}{2\pi}\, C_q^{\overline{\text{MS}}}\Big(\frac{x}{\xi}\Big) + \dots\right\}. \tag{1.4.17}$$

All modern fits, including ours, use $\overline{\text{MS}}$. A scheme does not change any prediction, provided the same one is used for the distributions and for every cross-section they are folded with. The explicit $C_q^{\overline{\text{MS}}}$ and its gluon counterpart are given in chapter 1.6.

6. The DGLAP equation

The scale μ was introduced by hand, so the left side of Eq. (1.4.16) cannot depend on it. Differentiate the right side with respect to $\ln \mu^2$ and set the result to zero. To first order in α_S the derivative of the distribution must cancel the derivative of the explicit logarithm:

$$\mu^2 \frac{\partial}{\partial \mu^2}\, q(x,\mu^2) = \frac{\alpha_S(\mu^2)}{2\pi} \int_x^1 \frac{d\xi}{\xi}\, P_{qq}\!\left(\frac{x}{\xi}\right) q(\xi,\mu^2). \tag{1.4.18}$$

This is the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) equation [3, 4]. Three points about it:

- **It is the analogue of the equation for the running coupling.** There, the bare coupling absorbs an ultraviolet divergence and the renormalised coupling runs with the scale. Here, the bare distribution absorbs a collinear divergence and the renormalised distribution runs with the scale.
- **The coupling is evaluated at μ^2.** The step above treated α_S as fixed. The full treatment, through the operator product expansion and the renormalisation group, confirms the equation with the running coupling in this position.
- **It is an initial-value problem.** Given $q(x,\mu_0^2)$ for all x at one scale, the equation gives it at every other scale. Only values $\xi \geq x$ enter: a quark at x can only have come from a parent with more momentum.

The integral is a convolution, written

$$(P \otimes q)(x) = \int_x^1 \frac{d\xi}{\xi}\, P\!\left(\frac{x}{\xi}\right) q(\xi), \qquad \text{so that} \qquad \frac{\partial q}{\partial \ln \mu^2} = \frac{\alpha_S}{2\pi}\, P_{qq} \otimes q. \tag{1.4.19}$$

Reading it as a rate equation

Write out the plus prescription in Eq. (1.4.18):

$$\frac{\partial q(x)}{\partial \ln \mu^2} = \frac{\alpha_S}{2\pi}\, C_F \left[\int_x^1 \frac{dz}{z}\, \frac{1+z^2}{1-z}\, q\!\left(\frac{x}{z}\right) \;-\; q(x) \int_0^1 dz\, \frac{1+z^2}{1-z} \right].$$

The first term is a gain: quarks at higher momentum x/z radiate and arrive at x. The second is a loss: quarks at x radiate and leave. Each integral diverges at $z = 1$ and the difference is finite. As μ rises the photon resolves shorter distances, sees more of the radiation, and finds the quark's momentum shared with more gluons. $P_{qq}(z)$ is the probability density, per unit $\ln \mu^2$ and in units of $\alpha_S/2\pi$, for finding a quark with a fraction z of its parent's momentum.

7. Gluons, and the full set of splitting functions

Equation (1.4.18) is exact only for a difference of two quark distributions. In general, gluons feed the quarks.

The gluon-initiated process

At the same order in α_S the photon can scatter off a quark that came from a gluon, $\gamma^* g \to q\bar{q}$. The calculation runs exactly as in section 3. There is again a logarithm from vanishing quark virtuality:

$$\hat{F}_2^{\,g}(x,Q^2) = x\sum_{q,\bar{q}} e_q^2\,\frac{\alpha_S}{2\pi}\Big(P_{qg}(x)\,\ln\frac{Q^2}{\kappa^2} + C_g(x)\Big), \tag{1.4.20}$$

with the splitting function $P_{qg}(x) = T_R\big[x^2+(1-x)^2\big]$ and $T_R = \frac{1}{2}$. Folded with a bare gluon distribution g_0, this logarithm is absorbed into the quark distribution alongside the one from quark emission. So the quark distribution at scale μ depends on the gluon distribution, and its evolution acquires a term $P_{qg}\otimes g$.

The four kernels

A gluon distribution has its own equation, with a quark source ($q \to g\,q$, seen from the gluon's side) and a gluon self-coupling ($g \to gg$). At leading order the four splitting functions are [3]:

$$\begin{aligned}
P_{qq}(x) &= C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\,\delta(1-x)\right],\\
P_{qg}(x) &= T_R\big[x^2+(1-x)^2\big],\\
P_{gq}(x) &= C_F\,\frac{1+(1-x)^2}{x},\\
P_{gg}(x) &= 2C_A\left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x)\right] + \frac{11C_A - 4n_f T_R}{6}\,\delta(1-x).
\end{aligned} \tag{1.4.21}$$

$P_{ab}(x)$ is for finding parton a inside parton b with a fraction x of its momentum. $C_A = 3$ is the colour factor for a gluon emitting a gluon, and n_f is the number of quark flavours light enough to be produced.

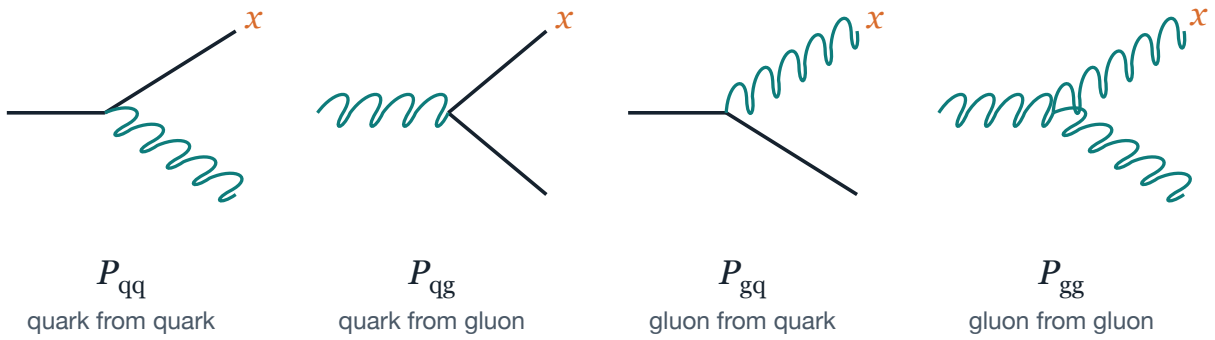

Figure 2. The four leading-order splittings. Straight lines are quarks, curly lines gluons. In each, the parent enters from the left and the daughter marked x is the one that goes on to the hard scattering. P_{qq} and P_{gq} are the same vertex read from its two daughters, which is why $P_{gq}(x) = \hat{P}_{qq}(1-x)$.

What fixes them

Only two of the four functions need a real calculation. The rest follows from conservation laws (derivation D4).

$$\begin{aligned} \int_0^1 dx\, P_{qq}(x) &= 0, \\ \int_0^1 dx\, x\,\big[P_{qq}(x) + P_{gq}(x)\big] &= 0, \\ \int_0^1 dx\, x\,\big[2n_f P_{qg}(x) + P_{gg}(x)\big] &= 0. \end{aligned} \tag{1.4.22}$$

The first is quark-number conservation, used above. The second says that when a quark splits, the momentum of the daughter quark and gluon adds up to the parent's. The third says the same for a gluon, which can split into a gluon pair or into any of n_f quark–antiquark pairs.

The coefficient of $\delta(1-x)$ in P_{gg} is fixed by the third sum rule. Its value, $(11C_A - 4n_f T_R)/6 = (33 - 2n_f)/6$, is 2π times the leading coefficient b of the QCD beta function, Eq. (1.4.27). The same gluon self-energy loop that makes the coupling run is the one that removes gluons at $x = 1$.

The matrix equation

Charge conjugation and flavour symmetry make P_{qg} and P_{gq} the same for every flavour and for quarks and antiquarks. Define the *singlet* distribution, the sum over all quarks and antiquarks:

$$\Sigma(x,\mu^2) = \sum_i \big[q_i(x,\mu^2) + \bar{q}_i(x,\mu^2)\big].$$

It mixes with the gluon:

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} \Sigma \\ g \end{pmatrix} = \frac{\alpha_S(\mu^2)}{2\pi} \begin{pmatrix} P_{qq} & 2n_f P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ g \end{pmatrix}. \tag{1.4.23}$$

Any *non-singlet* combination, one in which the gluon contribution cancels, evolves on its own by Eq. (1.4.18). Examples are the valence distributions $q_i - \bar{q}_i$ and flavour differences such as $u + \bar{u} - d - \bar{d}$.

This is why fitting codes do not work with $u, \bar{u}, d, \ldots$ directly. They use the *evolution basis*: Σ and g, which obey the coupled pair of equations, together with valence and flavour-difference combinations (V, V_3, V_8, T_3, T_8, T_{15}, ...) that each obey one uncoupled equation. In that basis a $(2n_f + 1)$-dimensional problem reduces to one 2×2 system and a list of independent equations. FK tables take their input in this basis, and Colibri's models return their distributions in it (chapter 2.1).

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8. Solving the equations: moments

A convolution becomes a product under a Mellin transform. Define the moments

$$q(N, \mu^2) = \int_0^1 dx\, x^{N-1}\, q(x, \mu^2), \qquad \gamma_{ab}(N) = \int_0^1 dx\, x^{N-1}\, P_{ab}(x). \tag{1.4.24}$$

Taking the N-th moment of Eq. (1.4.18) and substituting $x = z\,\xi$ separates the double integral:

$$\mu^2 \frac{\partial}{\partial \mu^2}\, q(N, \mu^2) = \frac{\alpha_S(\mu^2)}{2\pi}\, \gamma_{qq}(N)\, q(N, \mu^2). \tag{1.4.25}$$

An integro-differential equation in x has become an ordinary differential equation for each N. The numbers $\gamma_{ab}(N)$ are the *anomalous dimensions*. At leading order (derivation D5):

$$\begin{aligned} \gamma_{qq}(N) &= C_F \left[\frac{3}{2} + \frac{1}{N(N+1)} - 2 \sum_{k=1}^{N} \frac{1}{k} \right], \\ \gamma_{qg}(N) &= T_R\, \frac{2 + N + N^2}{N(N+1)(N+2)}, \qquad \gamma_{gq}(N) = C_F\, \frac{2 + N + N^2}{N(N^2-1)}, \\ \gamma_{gg}(N) &= 2C_A \left[\frac{11}{12} + \frac{1}{N(N-1)} + \frac{1}{(N+1)(N+2)} - \sum_{k=1}^{N} \frac{1}{k} \right] - \frac{2}{3}\, n_f T_R. \end{aligned} \tag{1.4.26}$$

The non-singlet solution

At leading order the coupling runs as

$$\alpha_S(\mu^2) = \frac{1}{b\,\ln(\mu^2/\Lambda^2)}, \qquad b = \frac{33 - 2n_f}{12\pi}, \tag{1.4.27}$$

so $d\alpha_S/d\ln\mu^2 = -b\,\alpha_S^2$. Dividing Eq. (1.4.25) by this turns $\ln\mu^2$ into α_S as the variable, $dq/d\alpha_S = -\gamma_{qq}\,q/(2\pi b\,\alpha_S)$, which integrates to

$$q(N,\mu^2) = q(N,\mu_0^2)\left[\frac{\alpha_S(\mu_0^2)}{\alpha_S(\mu^2)}\right]^{d_{qq}(N)}, \qquad d_{qq}(N) = \frac{\gamma_{qq}(N)}{2\pi b}. \tag{1.4.28}$$

Three consequences can be read off directly.

- $N = 1$**: the number of valence quarks.** $\gamma_{qq}(1) = 0$, so the first moment does not evolve. The proton has two valence up quarks and one down at every scale.
- $N \geq 2$**: momentum drains to small** x**.** $\gamma_{qq}(N)$ is negative for $N > 1$ and falls without limit, like $-2C_F \ln N$. High moments weight large x, so the large-x part of the distribution is depleted fastest. The distribution slides towards small x as μ rises.
- **The change is slow.** Moments vary as a power of α_S, that is, as a power of $\ln\mu^2$. Going from $\mu^2 = 10$ to 10^4 GeV^2 changes the ratio of couplings by a factor of about 2.2.

The singlet solution

For Σ and g, Eq. (1.4.23) becomes a 2×2 matrix equation for each N. Its eigenvalues are

$$\gamma_\pm(N) = \tfrac{1}{2}\Big[\gamma_{qq} + \gamma_{gg} \pm \sqrt{(\gamma_{qq} - \gamma_{gg})^2 + 8n_f\,\gamma_{qg}\gamma_{gq}}\Big], \tag{1.4.29}$$

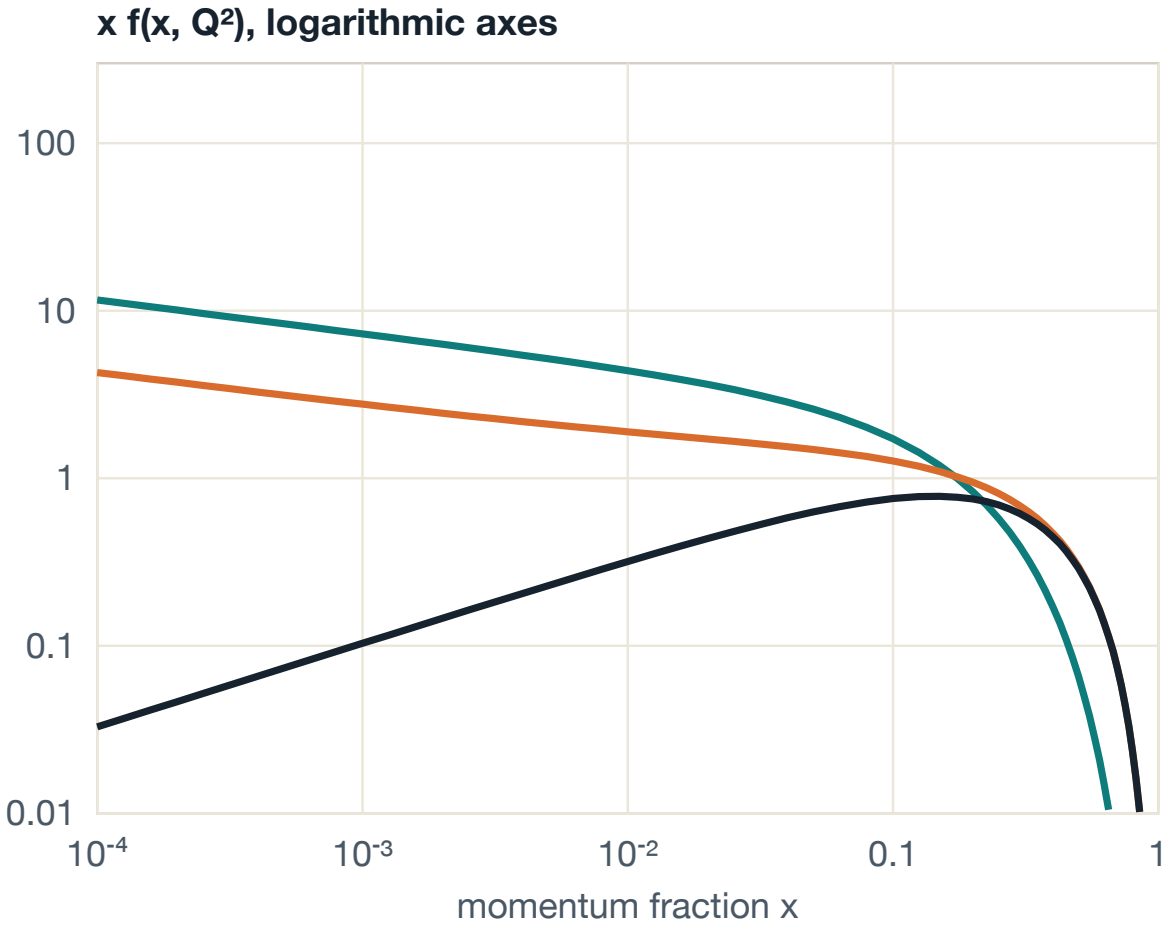
and each eigenvector evolves as in Eq. (1.4.28) with its own exponent $\gamma_\pm/2\pi b$.

The second moment is the total momentum. At $N = 2$ the eigenvalues are $\gamma_+ = 0$ and $\gamma_- = -\big(\tfrac{4}{3}C_F + \tfrac{1}{3}n_f\big)$. The zero eigenvalue is momentum conservation: $\Sigma(2) + g(2) = 1$ at all scales. The negative one decays, so as $\mu \to \infty$ the momentum settles into a fixed ratio:

$$\frac{\Sigma(2)}{g(2)} \;\longrightarrow\; \frac{3n_f}{16}, \qquad \text{so gluons carry } \frac{16}{16 + 3n_f}. \tag{1.4.30}$$

For $n_f = 5$ that is 52% for the gluons. The measured fraction at accessible scales is already close to one half (chapter 1.3), which is why half of the proton's momentum is invisible to the photon.

Leading-order evolution, solved exactly



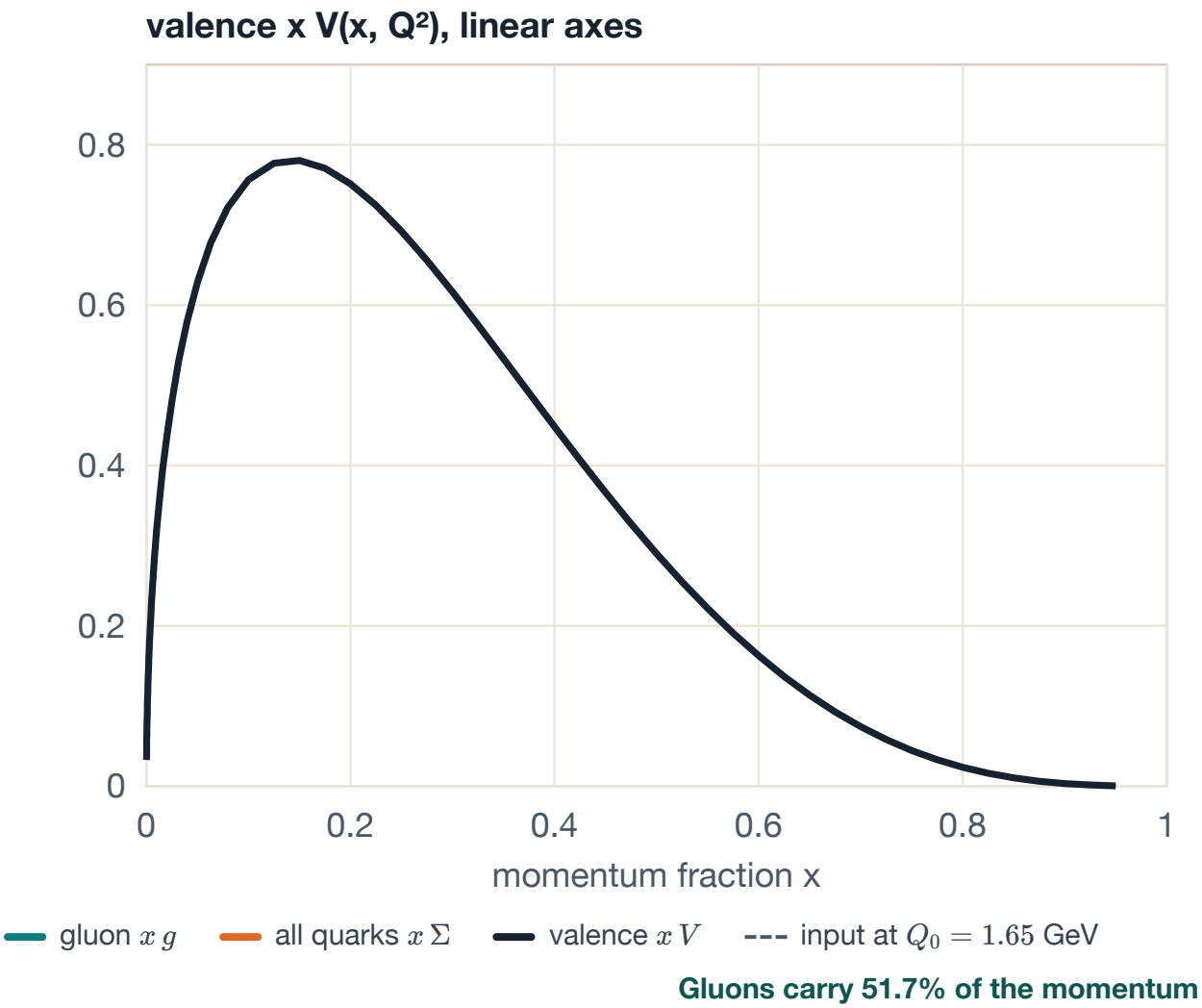


Figure 3. Move the slider to raise the scale. Left, on logarithmic axes: the gluon and the quark sea grow steeply at small x. Right, on linear axes: the valence distribution loses its large-x tail and its peak moves left, while its area divided by x (the number of valence quarks, 3) stays fixed. The curves are the exact solution of Eqs. (1.4.28) and (1.4.29) with $n_f = 4$ and $\Lambda = 0.2$ GeV, inverted numerically. The input shapes are illustrative, not a fit.

9. What the equation predicts

The pattern of scaling violation

At large x the quarks dominate and Eq. (1.4.18) applies. The loss term wins, so $\partial F_2/\partial \ln Q^2 < 0$. At small x the gluon term dominates the quark equation:

$$\frac{\partial F_2(x,Q^2)}{\partial \ln Q^2} \;\approx\; x\sum_{q,\bar{q}} e_q^2\,\frac{\alpha_S(Q^2)}{2\pi}\,\big(P_{qg}\otimes g\big)(x) \;>\; 0. \tag{1.4.31}$$

This is the pattern seen in the data: F_2 falls with Q^2 at large x and rises at small x. It also explains how deep inelastic scattering measures the gluon, which the photon does not couple to. The slope of F_2 in $\ln Q^2$ at small x is proportional to α_S times the gluon distribution.

The rise at small x

Two of the splitting functions are singular at small x: $P_{gq} \to 2C_F/x$ and $P_{gg} \to 2C_A/x$. A parton readily radiates a gluon carrying a small fraction of its momentum, and that gluon radiates again. In moment space the singularity is a pole, $\gamma_{gg}(N) \simeq 2C_A/(N-1)$. Keeping only this term and inverting the transform by the saddle-point method (derivation D6) gives

$$x\,g(x,\mu^2) \;\sim\; \exp\sqrt{\frac{4C_A}{\pi b}\;\ln\frac{1}{x}\;\ln\frac{\alpha_S(\mu_0^2)}{\alpha_S(\mu^2)}}. \tag{1.4.32}$$

The gluon grows faster than any power of $\ln(1/x)$ but slower than any power of $1/x$, and the growth strengthens with μ^2. This steep rise of F_2 at small x was observed at HERA from 1993. At very small x the leading-order kernel is not enough, because terms $\alpha_S^n \ln^n(1/x)$ from higher orders become as large; that regime is taken up in chapter 3.3.

What the equation sums

Solving Eq. (1.4.18) to all orders in α_S is not the same as computing one emission. Expand the solution for fixed coupling:

$$q(x,Q^2) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\alpha_S}{2\pi}\ln\frac{Q^2}{\mu_0^2}\right)^n \big(P_{qq}\otimes\big)^n q(x,\mu_0^2). \tag{1.4.33}$$

The n-th term is the emission of n gluons in succession. It carries n logarithms only when the transverse momenta are strongly ordered along the chain, $\mu_0^2 \ll k_{T1}^2 \ll k_{T2}^2 \ll \cdots \ll Q^2$, and the nested integrals over that region give the $1/n!$. When $\alpha_S \ln(Q^2/\mu_0^2)$ is of order one, every term is as important as the first. The DGLAP equation sums all of them: it is the *leading-logarithm* approximation, exact in the terms $\alpha_S^n \ln^n Q^2$.

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10. Beyond leading order, and where our fits use this

Higher orders

The splitting functions have an expansion in the coupling,

$$P_{ab}(x,\alpha_S) = P_{ab}^{(0)}(x) + \frac{\alpha_S}{2\pi}P_{ab}^{(1)}(x) + \left(\frac{\alpha_S}{2\pi}\right)^2 P_{ab}^{(2)}(x) + \ldots \tag{1.4.34}$$

and the equations keep the form of Eq. (1.4.23) at every order. The functions in Eq. (1.4.21) are $P^{(0)}$.

ORDER	SUMS	SPLITTING FUNCTIONS	STATUS
LO	$\alpha_S^n \ln^n Q^2$	$P^{(0)}$, 1972–77 [3, 4]	too imprecise for fits
NLO	$\alpha_S^n \ln^{n-1} Q^2$	$P^{(1)}$, 1980 [5]	scheme dependence appears
NNLO	$\alpha_S^n \ln^{n-2} Q^2$	$P^{(2)}$, 2004 [6]	the standard for global fits
N³LO	$\alpha_S^n \ln^{n-3} Q^2$	$P^{(3)}$, known in part	approximate fits since 2022 [7, 8]

From NLO on, the splitting functions depend on the factorisation scheme, quark and antiquark kernels differ, and three separate non-singlet kernels appear. The structure of section 7 survives: one coupled singlet system and a set of uncoupled equations.

In this project

Our fits use NNPDF theory 40000000: NNLO evolution and coefficient functions, $\alpha_S(M_Z) = 0.118$, starting scale $Q_0 = 1.65$ GeV. Three things in the fits follow directly from this chapter.

- **The parameters live at one scale.** The 52 parameters of the MSHT20 form describe the distributions at Q_0 only. Every data point, at whatever Q^2, is predicted by evolving from there.
- **Evolution is precomputed.** The evolution from Q_0 to each data point's scale is linear in the input distributions. It is done once and folded with the hard cross-section into an FK table, so a prediction is a single matrix product of the table with the distributions on a grid in x (chapter 2.1).
- **The gluon is constrained indirectly.** Deep inelastic data reach the gluon through Eq. (1.4.31), the Q^2 slope of F_2, and through the momentum sum rule. This is one reason the gluon and the sea are less directly constrained than the valence quarks. The directions our fits cannot pin down sit mostly in the strange and sea sector (Results, Phase A and Phase B).

Derivations

D1. The two-particle phase space in light-cone variables

Derives (1.4.9), (1.4.10)

The phase space for the gluon r and the outgoing quark l is

$$d\Phi_2 = \int \frac{d^4r}{(2\pi)^3}\frac{d^4l}{(2\pi)^3}\,\delta^+(r^2)\,\delta^+(l^2)\,(2\pi)^4\delta^4(p+q-r-l).$$

Use the four-dimensional delta function to do the l integral, and change variable from r to $k = p - r$:

$$d\Phi_2 = \frac{1}{4\pi^2}\int d^4k\;\delta^+\big((p-k)^2\big)\,\delta^+\big((k+q)^2\big).$$

The measure. Write $k = a\,p + b\,n + k_T$. Because $n\cdot p = 1$, the measure is $d^4k = da\,db\,d^2k_T$, and $k^2 = 2ab - k_T^2$. Setting $a = \xi$ and trading b for k^2 gives $db = dk^2/2\xi$, which is the parametrisation of Eq. (1.4.6). With $d^2k_T = \frac{1}{2}\,dk_T^2\,d\theta$:

$$d^4k = \frac{d\xi}{4\xi}\;d|k^2|\;dk_T^2\;d\theta, \qquad 0<\theta<2\pi.$$

The two mass-shell conditions. From Eq. (1.4.6), $p\cdot k = (k_T^2 - |k^2|)/2\xi$ and $q\cdot k = \xi\nu - q_T\cdot k_T$, where the last product is the ordinary two-dimensional one. Then

$$(p-k)^2 = -2p\cdot k + k^2 = \frac{(1-\xi)|k^2| - k_T^2}{\xi}, \qquad (k+q)^2 = 2\xi\nu - Q^2 - |k^2| - 2\,q_T\cdot k_T.$$

Setting both to zero gives Eq. (1.4.7). As delta functions,

$$\delta\big((p-k)^2\big) = \xi\,\delta\big(k_T^2 - (1-\xi)|k^2|\big), \qquad \delta\big((k+q)^2\big) = \frac{1}{2\nu}\,\delta\Big(\xi - x - \frac{|k^2| + 2\,q_T\cdot k_T}{2\nu}\Big).$$

Assembling. The factors of ξ cancel. The integrand depends on θ only through $\cos\theta$, so the range $0<\theta<2\pi$ is twice the range $0<\theta<\pi$:

$$d\Phi_2 = \frac{1}{16\pi^2\nu}\int d\xi\;d|k^2|\;dk_T^2\int_0^\pi d\theta\;\delta\big(k_T^2-(1-\xi)|k^2|\big)\;\delta\Big(\xi - x - \frac{|k^2| + 2\,q_T\cdot k_T}{2\nu}\Big).$$

The structure function. By Eq. (1.4.4) and the normalisation used for Eq. (1.4.5), $\hat F_2 = \nu\int d\Phi_2\,\frac{1}{4\pi}n_\alpha n_\beta\overline{\sum}|\mathcal{M}|^2$. Insert Eq. (1.4.8). The k_T^2 integral is done by the first delta function. For the angle, write $q_T\cdot k_T = Q\,k_T\cos\theta$ and $z = |k^2|/2\nu$; with $Q^2 = 2\nu x$ and $k_T^2 = 2\nu z(1-\xi)$ the second delta function is $\delta(A - B\cos\theta)$ with

$$A = \xi - x - z, \qquad B = 2\sqrt{xz(1-\xi)}, \qquad \int_0^\pi d\theta\;\delta(A - B\cos\theta) = \frac{1}{\sqrt{B^2 - A^2}}.$$

$B^2 - A^2$ is a quadratic in ξ: $B^2 - A^2 = -\left[\xi^2 - 2\xi(x + z - 2xz) + (x - z)^2\right] = (\xi_+ - \xi)(\xi - \xi_-)$, with roots

$$\xi_\pm = x + z - 2xz \pm \sqrt{(x + z - 2xz)^2 - (x - z)^2} = x + z - 2xz \pm \sqrt{4x(1-x)z(1-z)}.$$

The roots are real only for $0 < z < 1$, which sets the upper limit $|k^2| < 2\nu$. Collecting factors, $\nu \cdot \frac{1}{16\pi^2\nu} \cdot 8e_q^2\alpha_S = e_q^2\alpha_S/2\pi^2$, which is Eq. (1.4.9).

The limit. Substituting $\xi = \frac{1}{2}(\xi_+ + \xi_-) + \frac{1}{2}(\xi_+ - \xi_-)\sin u$ gives $\int_{\xi_-}^{\xi_+} d\xi / \sqrt{(\xi_+ - \xi)(\xi - \xi_-)} = \int_{-\pi/2}^{\pi/2} du = \pi$ for any z. As $z \to 0$ both roots tend to x, so the smooth factor $\xi\hat{P}(\xi)$ can be taken out at $\xi = x$. This gives Eq. (1.4.10).

D2. The projected matrix element and the splitting function

Derives (1.4.8)

Squaring. Average over the two spin states and three colours of the incoming quark and sum over everything else. The colour sum is $\frac{1}{3}\,\mathrm{Tr}(t^A t^A) = C_F$. With massless spinors, $\sum u\bar{u} = \not{p}$:

$$\overline{\sum}|\mathcal{M}|^2_{\alpha\beta} = \frac{1}{2}\,e_q^2\,g^2\,C_F \sum_{\text{pol}} \frac{1}{k^4}\,\mathrm{Tr}\left[\gamma_\beta\,\not{l}\gamma_\alpha\,\not{k}\not{\varepsilon}\not{p}\not{\varepsilon}^*\not{k}\right], \qquad l = k + q.$$

Projecting onto F_2. Contract with $n^\alpha n^\beta$. Because $n^2 = 0$, $\not{n}\not{l}\not{n} = 2(n\cdot l)\,\not{n}$. From Eqs. (1.4.3) and (1.4.6), $n \cdot q = 0$ and $n \cdot k = \xi$, so $n \cdot l = \xi$ and the trace becomes $2\xi\,\mathrm{Tr}\left[\not{n}\not{k}\not{\varepsilon}\not{p}\not{\varepsilon}^*\not{k}\right]$.

The polarisation sum. A real gluon has two transverse polarisations. In the light-cone gauge $n \cdot \varepsilon = 0$ the sum over them is

$$\sum_{\text{pol}} \varepsilon_\mu(r)\,\varepsilon_\nu^*(r) = -g_{\mu\nu} + \frac{n_\mu r_\nu + n_\nu r_\mu}{n\cdot r}, \qquad n\cdot r = 1 - \xi.$$

Apply it to $\not{\varepsilon}\not{p}\not{\varepsilon}^*$. The first term gives $-\gamma^\mu\not{p}\gamma_\mu = 2\not{p}$. In the second, $r = p - k$ and $\not{p}\not{p} = p^2 = 0$, so $\not{n}\not{p}\not{r} = -\not{n}\not{p}\not{k}$ and $\not{r}\not{p}\not{n} = -\not{k}\not{p}\not{n}$:

$$\sum_{\text{pol}} \not{\varepsilon}\not{p}\not{\varepsilon}^* = 2\not{p} - \frac{\not{n}\not{p}\not{k} + \not{k}\not{p}\not{n}}{1 - \xi}.$$

The two traces. Use $\not{k}\not{k} = k^2$ and $p \cdot k = k^2/2$, which is the gluon's mass-shell condition $(p - k)^2 = 0$.

For the first term, $\not{k}\not{p}\not{k} = 2(p\cdot k)\not{k} - k^2\not{p}$, so

$$2\,\mathrm{Tr}\big[\not n\,\not k\not p\not k\big] = 8\big[2(p\cdot k)(n\cdot k) - k^2\,(n\cdot p)\big] = 8\,k^2(\xi-1) = 8\,(1-\xi)\,|k^2|.$$

For the second, each piece reduces to $k^2\,\mathrm{Tr}[\not n\not k\not n\not p] = k^2\cdot 4\cdot 2(n\cdot k)(n\cdot p) = 8\xi k^2$, and there are two of them:

$$-\frac{16\,\xi\,k^2}{1-\xi} = \frac{16\,\xi\,|k^2|}{1-\xi}.$$

The sum.

$$\mathrm{Tr}\Big[\not n\not k\Big(\sum_{\mathrm{pol}} \not\varepsilon\not p\not\varepsilon^*\Big)\not k\Big] = 8\,|k^2|\,\frac{(1-\xi)^2+2\xi}{1-\xi} = 8\,|k^2|\,\frac{1+\xi^2}{1-\xi}.$$

One power of $|k^2|$ has appeared in the numerator and cancels one of the two propagators. Collecting factors,

$$n^\alpha n^\beta\,\overline{\sum}|\mathcal{M}|^2_{\alpha\beta} = \frac{1}{2}\,e_q^2 g^2 C_F\cdot 2\xi\cdot\frac{8}{|k^2|}\,\frac{1+\xi^2}{1-\xi} = \frac{8\,e_q^2\,g^2}{|k^2|}\;\xi\;C_F\frac{1+\xi^2}{1-\xi},$$

and with $g^2 = 4\pi\alpha_S$ this is Eq. (1.4.8). The numerator $1+\xi^2 = (1-\xi)^2 + 2\xi$ has a direct reading: $(1-\xi)^2$ comes from the $-g_{\mu\nu}$ term and 2ξ from the gauge term that removes the unphysical polarisations.

The check script evaluates this trace with explicit 4×4 Dirac matrices at two kinematic points and reproduces $8|k^2|(1+\xi^2)/(1-\xi)$ to better than one part in 10^8.

D3. The virtual term from quark-number conservation

Derives (1.4.12)

Write the complete splitting function as the real-emission part, regulated at $x=1$ by the plus prescription, plus an unknown multiple of $\delta(1-x)$:

$$P_{qq}(x) = C_F\left[\frac{1+x^2}{(1-x)_+} + K\,\delta(1-x)\right].$$

Take the first moment of Eq. (1.4.18). By the argument that leads to Eq. (1.4.25), $\frac{d}{d\ln\mu^2}\int_0^1 q\,dx = \frac{\alpha_S}{2\pi}\big(\int_0^1 P_{qq}\,dx\big)\big(\int_0^1 q\,dx\big)$. The number of quarks minus antiquarks is conserved, so $\int_0^1 P_{qq}\,dx = 0$. Using the definition (1.4.13) with $f(x) = 1+x^2$, $f(1)=2$:

$$\int_0^1 dx\,\frac{1+x^2}{(1-x)_+} = \int_0^1 dx\,\frac{x^2-1}{1-x} = -\int_0^1 dx\,(1+x) = -\frac{3}{2}.$$

Hence $K = \frac{3}{2}$, which is Eq. (1.4.12). The direct calculation of the quark self-energy and vertex diagrams gives the same coefficient.

D4. The other splitting functions from symmetry and sum rules

Derives (1.4.21), (1.4.22)

P_{gq} from P_{qq}. When a quark splits into a quark with fraction x and a gluon, the gluon has fraction $1-x$. For $x<1$ the probability of finding the gluon at x is therefore that of finding the quark at $1-x$:

$$P_{gq}(x) = \hat{P}_{qq}(1-x) = C_F\,\frac{1+(1-x)^2}{x}.$$

Momentum in quark splitting. With $f(x)=x(1+x^2)$, $f(1)=2$, and $x^3+x-2=(x-1)(x^2+x+2)$:

$$\int_0^1 dx\,x\,P_{qq}(x) = C_F\Big[-\int_0^1 (x^2+x+2)\,dx + \frac{3}{2}\Big] = C_F\Big[-\frac{17}{6}+\frac{3}{2}\Big] = -\frac{4}{3}\,C_F,$$

$$\int_0^1 dx\,x\,P_{gq}(x) = C_F\int_0^1 \big[1+(1-x)^2\big]dx = \frac{4}{3}\,C_F.$$

The sum is zero, the second relation of Eq. (1.4.22). The quark loses on average exactly the momentum the gluon gains.

The delta term in P_{gg}. The real-emission part of P_{gg} comes from the three-gluon vertex and is symmetric under $x\leftrightarrow 1-x$, since the two daughters are identical. Write $P_{gg} = 2C_A[\dots] + B\,\delta(1-x)$ and impose the third sum rule:

$$\int_0^1 dx\,x\cdot 2n_f P_{qg} = 2n_f T_R\Big(\frac{1}{4}+\frac{1}{12}\Big) = \frac{2}{3}\,n_f T_R,$$

$$\int_0^1 dx\,x\cdot 2C_A\Big[\frac{x}{(1-x)_+}+\frac{1-x}{x}+x(1-x)\Big] = 2C_A\Big[-\frac{3}{2}+\frac{1}{2}+\frac{1}{12}\Big] = -\frac{11}{6}\,C_A.$$

So $B = \frac{11}{6}C_A - \frac{2}{3}n_f T_R = (11C_A - 4n_f T_R)/6$, as in Eq. (1.4.21).

D5. The anomalous dimensions

Derives (1.4.26), (1.4.30)

All four follow from two elementary integrals: $\int_0^1 x^m dx = 1/(m+1)$ and $\int_0^1 \frac{1-x^m}{1-x} dx = \sum_{k=1}^{m} \frac{1}{k} \equiv H_m$.

γ_{qq}. With $f(x) = x^{N-1}(1+x^2)$ and $f(1) = 2$,

$$\int_0^1 dx\, \frac{x^{N-1} + x^{N+1} - 2}{1-x} = -H_{N-1} - H_{N+1} = -2H_N + \frac{1}{N} - \frac{1}{N+1} = -2H_N + \frac{1}{N(N+1)}.$$

Adding the $\frac{3}{2}$ from the delta function gives $\gamma_{qq}(N)$ in Eq. (1.4.26). At $N = 1$: $\frac{3}{2} + \frac{1}{2} - 2 = 0$.

γ_{qg} and γ_{gq}. These are sums of powers:

$$\gamma_{qg} = T_R\Big[\frac{1}{N} - \frac{2}{N+1} + \frac{2}{N+2}\Big] = T_R\,\frac{N^2+N+2}{N(N+1)(N+2)},$$
$$\gamma_{gq} = C_F\Big[\frac{2}{N-1} - \frac{2}{N} + \frac{1}{N+1}\Big] = C_F\,\frac{N^2+N+2}{N(N^2-1)}.$$

γ_{gg}. The plus term gives $\int_0^1 \frac{x^N-1}{1-x} dx = -H_N$. The other two give $\frac{1}{N-1} - \frac{1}{N}$ and $\frac{1}{N+1} - \frac{1}{N+2}$. Using $-\frac{1}{N} + \frac{1}{N+1} = -\frac{1}{N(N+1)}$ and regrouping, with the delta term $\frac{11}{6}C_A - \frac{2}{3}n_f T_R$ added:

$$\gamma_{gg}(N) = 2C_A\Big[\frac{11}{12} + \frac{1}{N(N-1)} + \frac{1}{(N+1)(N+2)} - H_N\Big] - \frac{2}{3}\,n_f T_R.$$

Checks at $N = 2$. $\gamma_{qq} = -\frac{4}{3}C_F$, $\gamma_{gq} = +\frac{4}{3}C_F$, $2n_f\gamma_{qg} = \frac{1}{3}n_f$, $\gamma_{gg} = -\frac{1}{3}n_f$. Each column of the matrix in Eq. (1.4.23) sums to zero, which is momentum conservation. The matrix $\begin{pmatrix} -\frac{4}{3}C_F & \frac{1}{3}n_f \\ \frac{4}{3}C_F & -\frac{1}{3}n_f \end{pmatrix}$ has eigenvalues 0 and $-(\frac{4}{3}C_F + \frac{1}{3}n_f)$, and its zero mode has $\Sigma/g = \frac{1}{3}n_f / \frac{4}{3}C_F = 3n_f/16$, which is Eq. (1.4.30).

D6. The double-logarithmic rise at small x

Derives (1.4.32)

Near $N = 1$ keep only the pole, $\gamma_{gg}(N) \simeq 2C_A/(N-1)$, and ignore the quarks. By Eq. (1.4.28) with $\zeta = \ln\big[\alpha_S(\mu_0^2)/\alpha_S(\mu^2)\big]$:

$$g(N,\mu^2) \simeq g(N,\mu_0^2)\,\exp\Big[\frac{C_A}{\pi b}\,\frac{\zeta}{N-1}\Big].$$

The inverse Mellin transform is $x\,g(x) = \frac{1}{2\pi i}\int dN\, x^{-(N-1)} g(N)$, along a line to the right of $N = 1$. With $\omega = N - 1$ and $L = \ln(1/x)$, the integrand is $\exp[h(\omega)]$ with

$$h(\omega) = \omega L + \frac{C_A}{\pi b}\,\frac{\zeta}{\omega}.$$

For large L the integral is dominated by the stationary point $h'(\omega_0) = 0$:

$$\omega_0 = \sqrt{\frac{C_A\,\zeta}{\pi b\,L}}, \qquad h(\omega_0) = 2\sqrt{\frac{C_A}{\pi b}\,\zeta\,L} = \sqrt{\frac{4C_A}{\pi b}\,\zeta\,L}.$$

This is Eq. (1.4.32). The result assumes the input $g(N, \mu_0^2)$ has no singularity to the right of ω_0, that is, the input gluon is no steeper than the generated one. For $n_f = 4$, $C_A/\pi b = 36/25$, so the exponent is $2.4\sqrt{\zeta\,L}$.

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