1.3 The parton model from field theory

The handbag diagram, the operator definition of a distribution, sum rules, and the infinite-momentum frame.

1. What is assumed and what is derived

Chapter 1.2 obtained Bjorken scaling, $F_2(x) = x \sum_q e_q^2\, q(x)$, from a picture: the proton is a beam of free point-like constituents, and the photon strikes one of them. That picture was proposed before QCD existed [3]. It contains four separate assumptions.

1. **Incoherence.** The photon scatters off one constituent at a time. Cross-sections for different constituents add; amplitudes do not interfere.
2. **Freedom.** During the scattering the struck constituent does not interact with the rest of the proton.
3. **Collinearity.** The constituent carries a fraction ξ of the proton's momentum and negligible transverse momentum and virtuality.
4. **Spin.** The charged constituents are point-like and have spin ½.

None of these is obviously true of a strongly bound state. This chapter rebuilds the model inside quantum field theory, so that each statement is either derived or isolated as a precise assumption. The result is the machinery chapter 1.4 starts from: the light-cone vectors p and n, the projection $F_2 = \nu\, n_\alpha n_\beta W^{\alpha\beta}$, and a definition of the quark distribution as the matrix element of an operator.

STATEMENT	STATUS HERE
The hadronic tensor is a correlator of two currents	derived, exact, Eq. (1.3.4)
The Bjorken limit probes the light cone $z^2 \to 0$	derived, Eq. (1.3.8)
The photon strikes a single quark (the handbag diagram)	assumed
The quark–proton amplitude $B(k, p)$ is damped at large transverse momentum and virtuality	assumed; false in QCD
Scaling of F_2	derived from the two assumptions, Eq. (1.3.19)
Callan–Gross relation $F_2 = 2xF_1$	derived, from spin ½, Eq. (1.3.22)
$q(x)$ is a matrix element of a light-cone operator	derived, Eq. (1.3.26)
Number and momentum sum rules	derived from conservation laws, Eqs. (1.3.33) and (1.3.35)
$q(x)$ is a number density	derived, in light-cone gauge only, Eq. (1.3.28)
The function $q(x)$ itself	not derivable in perturbation theory; measured

Why a fast proton helps

Assumptions 1 and 2 are statements about time scales, and they can be estimated. Take the proton to have a large momentum P along the z axis. Suppose it fluctuates into a quark with momentum fraction ξ and transverse

momentum k_T, plus a remainder with fraction $1-\xi$ and transverse momentum $-k_T$, both massless. The energy of the pair exceeds that of the proton by $\Delta E = k_T^2/2\xi(1-\xi)P$, so the fluctuation lives for a time $\tau_{\text{life}} \sim 1/\Delta E$. In the same frame the photon delivers an energy $q^0 = \nu/2P$ (section 4), so the absorption is resolved within $\tau_{\text{int}} \sim 1/q^0 = 4xP/Q^2$. This is the two-body case of the estimate in chapter 1.2, section 4. With $\xi \simeq x$ the ratio is

$$\frac{\tau_{\text{int}}}{\tau_{\text{life}}} \simeq \frac{2\,k_T^2}{(1-x)\,Q^2}. \tag{1.3.1}$$

For $k_T = 0.4$ GeV and $x = 0.3$ the ratio is 0.009 at $Q^2 = 50$ GeV^2 and 0.11 at $Q^2 = 4$ GeV^2. When $k_T^2 \ll Q^2$ the photon takes a snapshot of a frozen configuration, as an X-ray sees the atoms of a crystal at rest. The estimate also shows where the picture must fail: for fluctuations with $k_T^2 \sim Q^2$ (section 10).

The chapter follows Ellis, Stirling and Webber (ESW), section 4.2 [1], with the definitions and properties of the distributions from Moore's thesis, sections 1.1 and 1.2 [2].

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2. The hadronic tensor as a current correlator

This section shows that everything the photon measures is contained in the proton matrix element of a product of two electromagnetic currents.

From the amplitude to $W^{\alpha\beta}$

A lepton of momentum k scatters to momentum k' by exchanging a photon of momentum $q = k - k'$ with a proton of momentum P, producing a hadronic state X. To lowest order in the electromagnetic coupling e the amplitude is

$$\mathcal{A} = e^2\,\bar{u}(k')\gamma^{\alpha}u(k)\,\frac{1}{q^2}\,\langle X|\,j_{\alpha}(0)\,|P\rangle. \tag{1.3.2}$$

Here $j_\alpha = \sum_q e_q\,\bar{\psi}_q\gamma_\alpha\psi_q$ is the electromagnetic current of the quark fields ψ_q, with e_q the charge of flavour q in units of e. All the strong-interaction physics is in its matrix element between the proton and X.

The inclusive cross-section sums $|\mathcal{A}|^2$ over every state X. It factorises into a leptonic tensor, fixed by QED, contracted with the hadronic tensor

$$W_{\alpha\beta}(p,q) = \frac{1}{4\pi}\sum_X \langle P|\,j_\beta^{\dagger}(0)\,|X\rangle\langle X|\,j_\alpha(0)\,|P\rangle\,(2\pi)^4\delta^4(q+p-p_X). \tag{1.3.3}$$

The sum includes the phase-space integral of each state, and an average over the proton's spin is understood. The factor $1/4\pi$ is a convention; it is the "flux factor" used in chapter 1.4.

The correlator

The sum over states can be done exactly (derivation D1). Writing the delta function as a Fourier integral and using translation invariance and completeness, $\sum_X |X\rangle\langle X| = 1$:

$$\begin{aligned} W_{\alpha\beta}(p,q) &= \frac{1}{4\pi}\int d^4z\; e^{iq\cdot z}\,\langle P|\,j_\beta^{\dagger}(z)\,j_\alpha(0)\,|P\rangle \\ &= \frac{1}{4\pi}\int d^4z\; e^{iq\cdot z}\,\langle P|\,\big[j_\beta^{\dagger}(z),\,j_\alpha(0)\big]\,|P\rangle. \end{aligned} \tag{1.3.4}$$

The second form holds because the extra term in the commutator would need a state X with momentum $p - q$, which is lighter than the proton and does not exist. The commutator form matters for the next section: by causality, a commutator of two local operators vanishes when their separation is space-like.

Equation (1.3.4) is exact in the strong interaction. Deep inelastic scattering measures the Fourier transform of the correlation between the current at one point in the proton and the current a distance z away.

Structure functions

Current conservation, $q^\alpha W_{\alpha\beta} = 0$, and parity leave two independent tensors (chapter 1.1). Neglecting the proton mass, so that its momentum is a light-like vector p:

$$W^{\alpha\beta}(p,q) = \Big(-g^{\alpha\beta} + \frac{q^\alpha q^\beta}{q^2}\Big) F_1(x,Q^2) + \Big(p^\alpha + \frac{q^\alpha}{2x}\Big)\Big(p^\beta + \frac{q^\beta}{2x}\Big)\frac{F_2(x,Q^2)}{\nu}. \tag{1.3.5}$$

Here $\nu = p\cdot q$, $Q^2 = -q^2$ and $x = Q^2/2\nu$. This is ESW's Eq. (4.29) with $F_1 = W_1$ and $F_2 = \nu W_2$. ESW print the first bracket with the opposite overall sign; the sign used here is the one that reproduces their Eq. (4.35) and makes F_1 positive.

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3. Light-cone dominance

This section finds which separations z contribute to Eq. (1.3.4) in the Bjorken limit, $Q^2 \to \infty$ at fixed x. The answer is the neighbourhood of the light cone, $z^2 \lesssim 1/Q^2$. It is not the neighbourhood of the point $z = 0$.

The two components of the photon momentum

Work in the proton rest frame with the photon along $+z$. For any four-vector define the light-cone components $a^\pm = a^0 \pm a^3$. Then $a\cdot b = \tfrac{1}{2}(a^+b^- + a^-b^+) - a_T\cdot b_T$, and the photon has $q_T = 0$ and $q^+q^- = q^2 = -Q^2$. Its energy is $q^0 = \nu/M$, with M the proton mass, and its momentum is $q^3 = \sqrt{(q^0)^2 + Q^2}$. In the Bjorken limit (derivation D2)

$$q^+ \;\to\; \frac{2\nu}{M} \;\to\; \infty, \qquad q^- = -\frac{Q^2}{q^+} \;\to\; -Mx. \tag{1.3.6}$$

One component grows without limit. The other stays fixed.

Which z survive

The phase in Eq. (1.3.4) is $q\cdot z = \tfrac{1}{2}(q^+z^- + q^-z^+)$. Where the phase varies rapidly and the matrix element does not, the integral averages to zero. The integral is therefore dominated by

$$|z^-| \;\lesssim\; \frac{2}{q^+} = \frac{M}{\nu}, \qquad |z^+| \;\lesssim\; \frac{2}{|q^-|} = \frac{2}{Mx}. \tag{1.3.7}$$

Causality restricts the commutator to time-like or light-like separations, $z^2 = z^+z^- - z_T^2 \geq 0$. Combining this with Eq. (1.3.7):

$$0 \;\leq\; z^2 \;\leq\; z^+z^- \;\lesssim\; \frac{4}{q^+|q^-|} = \frac{4}{Q^2}. \tag{1.3.8}$$

As $Q^2 \to \infty$ the contributing region collapses onto the light cone $z^2 = 0$.

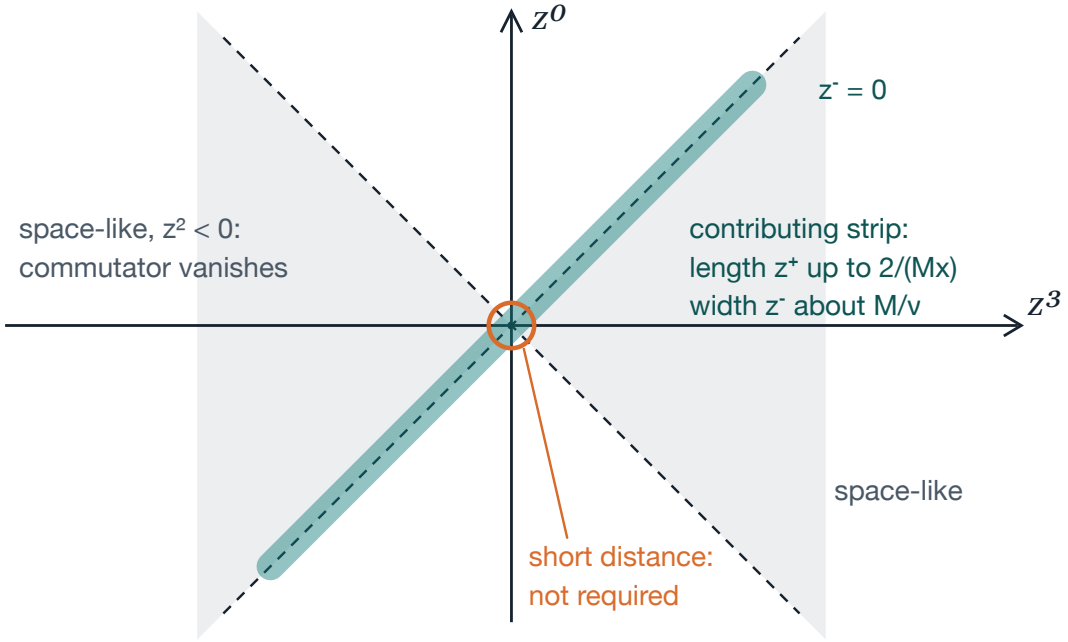

Figure 1. The separation z between the two currents, in the proton rest frame. The dashed lines are the light cone. The commutator vanishes in the shaded space-like regions. In the Bjorken limit the Fourier phase confines $z^- = z^0 - z^3$ to a width M/ν, which shrinks to zero, but leaves $z^+ = z^0 + z^3$ free up to $2/Mx$, which does not shrink. The orange circle marks the short-distance region, where every component of z is below $1/Q$; the strip extends far outside it.

Light cone, not short distance

The components of z are not all small. From Eq. (1.3.7), z^0 and z^3 are each of order $z^+/2 \sim 1/Mx$. In ordinary units

$$\frac{1}{Mx} = \frac{0.21\ \text{fm}}{x}: \qquad 2.1\ \text{fm at } x = 0.1, \qquad 21\ \text{fm at } x = 0.01. \tag{1.3.9}$$

At $x = 0.01$ the two currents are separated by many proton diameters along the direction of the photon, however large Q^2 is. The transverse separation, in contrast, is bounded by Eq. (1.3.8): $z_T^2 \lesssim 4/Q^2$.

The distinction has a consequence. If the dominant region were $z^\mu \to 0$, the product of currents could be replaced by a few local operators and the proton would contribute a few numbers. Because the dominant region is a line of finite length along the light cone, the proton contributes a function of the position along that line. Its Fourier transform is a function of x: the parton distribution (section 7).

The argument assumes a smooth matrix element. The commutator is in fact singular on the light cone, and that singularity is what survives. In a free-quark theory it comes from the quark propagator joining the two currents, which is the handbag diagram.

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4. Light-cone vectors and the projections

This section sets up the coordinates used in the rest of the guide and derives the two contractions that extract F_2 and F_L from the hadronic tensor.

The vectors p and n

The parton picture is clearest in a frame where the proton moves fast along $+z$. Neglect the proton mass and take two light-like vectors, p along the proton's momentum and n in the opposite direction:

$$p^\mu = (P, 0, 0, P), \qquad n^\mu = \Big(\frac{1}{2P}, 0, 0, -\frac{1}{2P}\Big), \qquad p^2 = n^2 = 0, \quad n\cdot p = 1. \tag{1.3.10}$$

Components are written (E, p_x, p_y, p_z). The vector n has dimensions of inverse mass. These are Eqs. (1.4.2) of chapter 1.4.

Decomposing any momentum

Any four-vector k can be written as a multiple of p, a multiple of n, and a remainder k_T orthogonal to both:

$$\begin{aligned} &k^\mu = (n\cdot k)\, p^\mu + (p\cdot k)\, n^\mu + k_T^\mu, \qquad n\cdot k_T = p\cdot k_T = 0, \\ &k^2 = 2\,(n\cdot k)(p\cdot k) - k_T^2. \end{aligned} \tag{1.3.11}$$

The coefficients follow by contracting with n and p in turn. The remainder k_T has only x and y components, and k_T^2 always denotes its positive squared length, which is why it enters k^2 with a minus sign. This is the Sudakov decomposition.

The coefficient $n\cdot k$ is the fraction of the proton's large momentum that k carries. In the frame of Eq. (1.3.10), $n\cdot k = (k^0 + k^3)/2P = k^+/p^+$. It is unchanged by boosts along z, which is why momentum fractions are defined this way.

The proton and the photon

With its mass restored, the proton's momentum is $P^\mu = p^\mu + \frac{1}{2}M^2 n^\mu$, which squares to M^2. The photon momentum is

$$q^\mu = \nu\, n^\mu + q_T^\mu, \qquad p\cdot q = \nu, \qquad n\cdot q = 0, \qquad q_T^2 = -q^2 = Q^2. \tag{1.3.12}$$

The condition $n\cdot q = 0$ is a choice: n is not fixed by p alone, and it is chosen so that the photon has no component along p. The photon's virtuality is then carried entirely by its transverse momentum. This frame differs from that of section 3, where $q_T = 0$, by a Lorentz transformation.

Projecting out F_2 and F_L

Contract Eq. (1.3.5) with $n_\alpha n_\beta$ and with $p_\alpha p_\beta$. Using $n^2 = p^2 = 0$, $n\cdot q = 0$, $n\cdot p = 1$ and $p\cdot q = \nu$ (derivation D3):

$$\nu\, n_\alpha n_\beta\, W^{\alpha\beta} = F_2, \qquad \frac{4x^2}{\nu}\, p_\alpha p_\beta\, W^{\alpha\beta} = F_2 - 2xF_1 \equiv F_L. \tag{1.3.13}$$

The first is Eq. (1.4.4) of chapter 1.4. The second defines the longitudinal structure function F_L, the response to longitudinally polarised photons.

5. The handbag diagram

This section states the parton-model assumptions as properties of one diagram and derives scaling and the Callan–Gross relation from them.

The assumption of incoherence

Suppose the two currents in Eq. (1.3.4) act on the same quark line, and the struck quark travels freely between them. Squared and summed over final states, this is the handbag diagram.

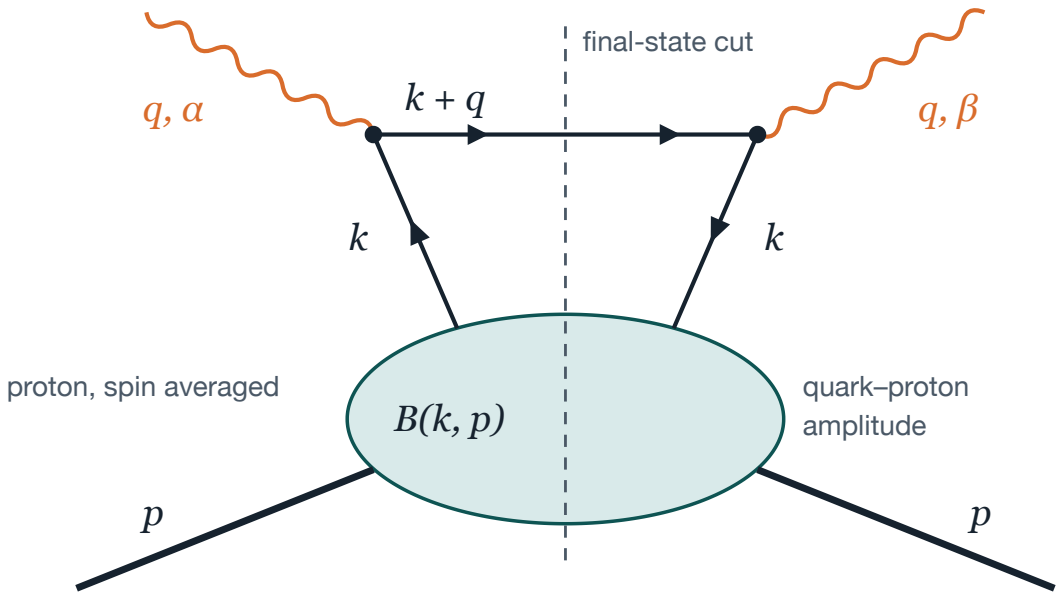

Figure 2. The handbag diagram. A quark of momentum k leaves the proton, absorbs the photon, and crosses the final-state cut (dashed) as a free on-shell quark of momentum $k+q$. The right half is the complex conjugate amplitude. The blob $B(k,p)$ contains everything about the proton: it is the amplitude for the proton to supply a quark of momentum k and take it back. The quark lines entering the blob are off shell.

For one flavour of charge e_q the diagram gives (derivation D4)

$$W^{\alpha\beta}(p,q) = e_q^2 \int \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\Big[\gamma^\alpha\,(\not{k}+\not{q})\,\gamma^\beta\, B(k,p)\Big]\,\delta\big((k+q)^2\big). \tag{1.3.14}$$

The factors are read off the figure: a vertex γ^α and its conjugate γ^β; the spin sum $\not{k}+\not{q}$ of the outgoing massless quark; the delta function that puts it on shell; and the blob

$$B_{ji}(k,p) = \frac{1}{2}\int d^4z\; e^{ik\cdot z}\,\langle P|\,\bar\psi_i(0)\,\psi_j(z)\,|P\rangle. \tag{1.3.15}$$

B is a 4×4 matrix in the Dirac indices i,j of the quark field ψ; a sum over colour is understood. It is the Fourier transform of the correlation between removing a quark at one point and replacing it at another. The factor $\frac{1}{2}$ is the product of the $1/4\pi$ in Eq. (1.3.3) and the 2π of the outgoing quark's phase space; it is included in B so that Eq. (1.3.14) agrees with ESW's Eq. (4.36).

The assumption is that the outgoing quark does not interact again with the remnant, and that diagrams in which the two photons attach to different quarks are negligible. Both are plausible when the time scales separate as in Eq. (1.3.1). Neither is proved here.

The assumption of damping

Decompose the quark momentum as in Eq. (1.3.11). Call its momentum fraction $\xi = n \cdot k$ and trade $p \cdot k$ for the virtuality k^2:

$$k^\mu = \xi\, p^\mu + \frac{k^2 + k_T^2}{2\xi}\, n^\mu + k_T^\mu, \qquad d^4k = \frac{d\xi}{2\xi}\, dk^2\, d^2k_T. \tag{1.3.16}$$

This is Eq. (1.4.6) of chapter 1.4, where $k^2 = -|k^2|$. The argument of the delta function in Eq. (1.3.14) is then exactly

$$(k+q)^2 = k^2 + 2\xi\nu - 2\, q_T \cdot k_T - Q^2,$$

with $q_T \cdot k_T$ the two-dimensional product. Now make the second assumption: $B(k,p)$ is strongly damped when k_T^2 or $|k^2|$ exceeds some hadronic scale m^2, of order the inverse proton size squared. Then in the region that matters, $k^2 \sim m^2$ and $q_T \cdot k_T \sim Qm$ are both small compared with Q^2, and

$$\delta\big((k+q)^2\big) \;\simeq\; \delta\big(2\xi\nu - Q^2\big) = \frac{1}{2\nu}\,\delta(\xi - x). \tag{1.3.17}$$

The neglected terms shift the position of the delta function by a relative amount of order m/Q, which averages to order m^2/Q^2 over the direction of k_T. Equation (1.3.17) gives the Bjorken variable its meaning: a massless quark can absorb the photon and stay massless only if its momentum fraction is $\xi = x$.

Scaling

Project out F_2 with Eq. (1.3.13). The Dirac algebra needs one identity. For any vector l, since $\not{n}\not{n} = n^2 = 0$,

$$\not{n}\not{l}\not{n} = 2\,(n\cdot l)\,\not{n} - \not{n}\not{n}\not{l} = 2\,(n\cdot l)\,\not{n}. \tag{1.3.18}$$

With $l = k + q$ and $n \cdot q = 0$, $n \cdot l = \xi$. Therefore

$$\begin{aligned} F_2 &= \nu\, e_q^2 \int \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\big[\not{n}\,(\not{k}+\not{q})\,\not{n}\,B\big]\, \frac{1}{2\nu}\,\delta(\xi - x) \\ &= e_q^2 \int \frac{d^4k}{(2\pi)^4}\, \xi\, \mathrm{Tr}\big[\not{n}\,B(k,p)\big]\,\delta(\xi - x) \;=\; e_q^2\, x\, q(x), \end{aligned} \tag{1.3.19}$$

where the quark distribution is

$$q(x) = \int \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\big[\not{n}\,B(k,p)\big]\,\delta(n\cdot k - x). \tag{1.3.20}$$

The factors of ν have cancelled. The right-hand side of Eq. (1.3.20) contains p, n and x and no reference to the photon, so F_2 depends on x alone. This is Bjorken scaling, now with a definition of $q(x)$ in place of a picture. Summing the handbag over flavours, and adding the diagram with the quark line reversed for antiquarks, gives $F_2 = x\sum_{q,\bar q} e_q^2\, q(x)$, Eq. (1.4.1).

For a free quark of momentum p, averaged over spin, $B = (2\pi)^4 \delta^4(k-p)\, \not{p}/4$ and Eq. (1.3.20) gives $q(x) = \delta(1-x)$. This reproduces $\hat{F}_2 = e_q^2\, \delta(1-x)$, Eq. (1.4.5).

The Callan–Gross relation

Project out F_L with the second contraction in Eq. (1.3.13). The same identity with p in place of n gives $\not{p}\not{l}\not{p} = 2(p\cdot l)\,\not{p}$, and $p\cdot l = \nu + (k^2 + k_T^2)/2\xi$. To leading order in $1/\nu$ (derivation D5):

$$F_L = \frac{4x^2}{\nu}\; e_q^2 \int \frac{d^4k}{(2\pi)^4}\; \mathrm{Tr}\big[\not{p}\, B(k,p)\big]\; \delta(\xi - x). \tag{1.3.21}$$

The integral does not depend on ν. It has dimensions of mass squared, and the only scale available is the damping scale m^2 of B. Hence $F_L \sim x^2 m^2/\nu \sim m^2/Q^2$, which vanishes in the Bjorken limit, and

$$F_L = F_2 - 2xF_1 = O\Big(\frac{m^2}{Q^2}\Big) \quad\Longrightarrow\quad F_2 = 2xF_1. \tag{1.3.22}$$

This is the Callan–Gross relation. Its origin is visible in the two traces. The vector n picks out the large component of the quark's momentum, $\mathrm{Tr}[\not{n}\not{k}] = 4\xi$. The vector p picks out the small one, $\mathrm{Tr}[\not{p}\not{k}] = 2(k^2 + k_T^2)/\xi$. Physically, a massless spin-½ quark collinear with the proton cannot absorb a longitudinal photon: in the frame where the quark reverses direction, the vector coupling conserves its helicity, so its spin projection changes by one unit, and a longitudinal photon supplies none.

The relation tests the spin of the constituents. For spin-0 constituents the vertex is $(2k+q)^\alpha$ in place of the Dirac trace, and the same two projections give $F_L = F_2$, that is $F_1 = 0$ (derivation D5). The data are close to $F_2 = 2xF_1$.

> **The two assumptions.** Scaling and the Callan–Gross relation follow from the handbag diagram plus one property of the blob: $B(k,p)$ falls off at large k_T^2 and $|k^2|$ fast enough for the integrals in Eqs. (1.3.20) and (1.3.21) to converge. Every Q^2-dependent correction is then a power of m^2/Q^2. Whether the integrals converge is a question about the theory. In QCD they do not.

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6. A worked model: when damping holds and when it fails

The handbag formulas can be evaluated exactly in a simple field theory. Doing so shows the size of the corrections dropped in Eq. (1.3.17), and what replaces scaling when B is not damped. The model is illustrative, not a description of the proton.

The model

Let the proton turn into a quark of momentum k and a single scalar particle of mass M_s, the spectator, which carries the remaining momentum $p-k$ into the final state. The vertex has a strength $g(k^2)$ that may depend on the quark's virtuality. The blob is then one tree diagram (derivation D6):

$$B(k,p) = \frac{\pi}{2}\; \big|g(k^2)\big|^2\; \frac{\not{k}\not{p}\not{k}}{(k^2)^2}\; \delta\big((p-k)^2 - M_s^2\big). \tag{1.3.23}$$

The delta function puts the spectator on shell and fixes the virtuality: $k^2 = -(k_T^2 + \xi M_s^2)/(1-\xi)$. The quark is space-like, and more so the larger its transverse momentum, as in Eq. (1.4.7). Inserting Eq. (1.3.23) in Eq. (1.3.20), with $\mathrm{Tr}[\not{n}\not{k}\not{p}\not{k}] = 4k_T^2$:

$$q(x) = \frac{1-x}{16\pi^2} \int_0^\infty dk_T^2 \, \frac{k_T^2 \, |g(k^2)|^2}{\left(k_T^2 + x M_s^2\right)^2}. \tag{1.3.24}$$

Two vertices

Everything depends on how $|g|^2$ behaves at large virtuality.

- **Damped vertex.** Take $|g(k^2)|^2 = g_0^2 \, \Lambda^4/(\Lambda^2 + |k^2|)^2$, which cuts off virtualities above Λ^2. The integrand of Eq. (1.3.24) falls as dk_T^2/k_T^6 and $q(x)$ is finite, as section 5 assumed.
- **Point-like vertex.** Take $|g|^2 = g_0^2$, a constant, as for a fundamental coupling in a renormalisable theory. The integrand falls only as dk_T^2/k_T^2. The integral diverges logarithmically, and $q(x)$ as defined by Eq. (1.3.20) does not exist.

The structure function itself is finite in both cases, because the exact delta function in Eq. (1.3.14) limits k_T^2 to values below about Q^2. Figure 3 shows the exact F_2 of the model, computed without the approximation (1.3.17).

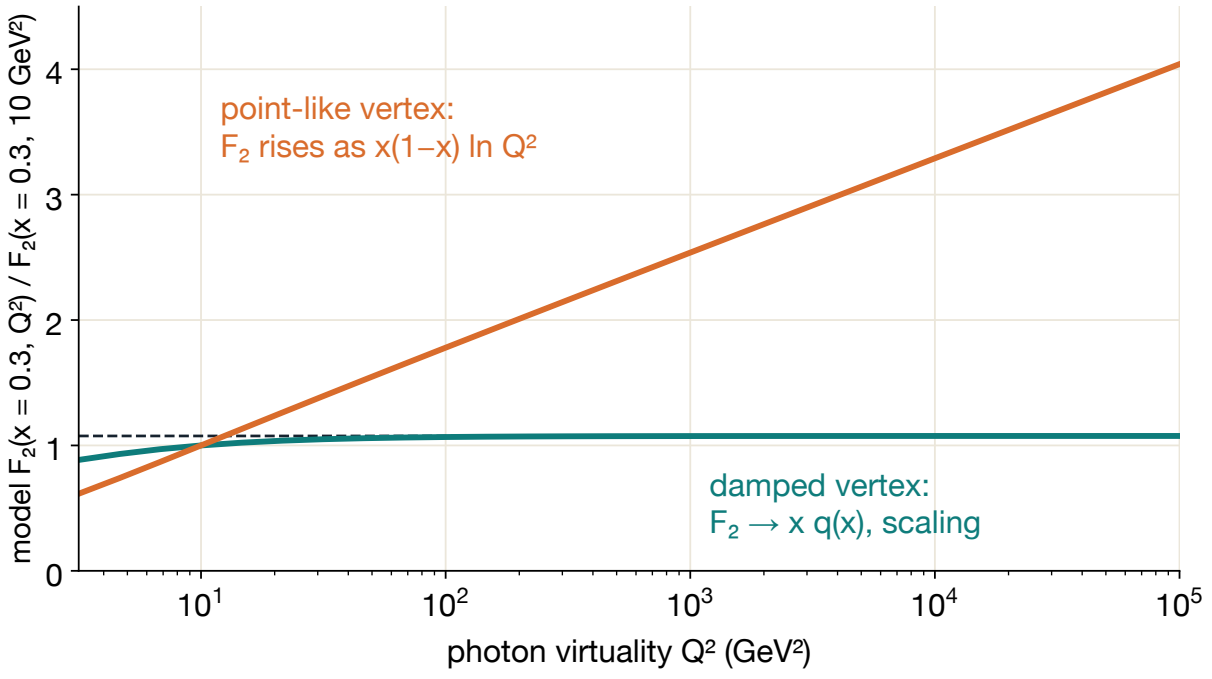

Figure 3. The exact handbag F_2 of the spectator model at $x = 0.3$, with $M_s = \Lambda = 0.8$ GeV, divided by its value at $Q^2 = 10\ \mathrm{GeV}^2$. Teal: damped vertex. The curve approaches the scaling limit $x\,q(x)$ (dashed); it is 7% below it at $10\ \mathrm{GeV}^2$, 0.8% at $100\ \mathrm{GeV}^2$ and 0.008% at $10^4\ \mathrm{GeV}^2$, a power correction. Orange: point-like vertex. F_2 grows by a factor of 4.0 between 10 and $10^5\ \mathrm{GeV}^2$ and never scales. Model curves, computed by the check script; not data.

For the damped vertex $Q^2 F_L/F_2 \to 3.25\ \mathrm{GeV}^2$ with these parameters: F_L is a power correction, as Eq. (1.3.22) states. For the point-like vertex,

$$\frac{\partial F_2}{\partial \ln Q^2} \;\to\; e_q^2\,\frac{g_0^2}{16\pi^2}\;x\,(1-x), \qquad F_L \;\to\; e_q^2\,\frac{g_0^2}{16\pi^2}\;2x^2. \tag{1.3.25}$$

Scaling is broken by a logarithm whose coefficient is a fixed function of x, and F_L is of the order of the coupling instead of m^2/Q^2.

This is the situation in QCD. The quark–gluon coupling is point-like, the probability to find a quark at transverse momentum k_T falls as $\alpha_S\,dk_T^2/k_T^2$, and the function of x multiplying the logarithm is the splitting function of chapter 1.4. The point-like spectator model is the same calculation with a scalar in place of the gluon, and $x(1-x)$ is its splitting function.

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7. The operator definition of a distribution

This section turns Eq. (1.3.20) into a statement about quark fields in position space. The result is the definition of a parton distribution used in all modern work.

The light-cone correlator

Insert Eq. (1.3.15) in Eq. (1.3.20). The integrals over $p\cdot k$ and k_T set $n\cdot z = 0$ and $z_T = 0$, leaving a separation $z = \lambda n$ along the light-like direction n (derivation D7):

$$q(x) = \int_{-\infty}^{\infty} \frac{d\lambda}{4\pi}\; e^{i\lambda x}\;\langle P|\,\bar\psi(0)\,\not n\,\psi(\lambda n)\,|P\rangle. \tag{1.3.26}$$

The distribution is the Fourier transform, in the light-cone distance λ, of the amplitude to remove a quark at λn and put it back at the origin. The momentum fraction x is conjugate to that distance, so small x corresponds to large λ. This is ESW's Eq. (4.42). ESW write $1/2\pi$ where Eq. (1.3.26) has $1/4\pi$; the factor of 2 is a matter of the spin sums and state normalisation, which ESW leave implicit (their footnote to Eq. 4.25). The $1/4\pi$ is also the normalisation of Collins and Soper [4]. With a spin-averaged proton state normalised to $2E$ particles per unit volume, $1/4\pi$ gives $q(x) = \delta(1-x)$ for a free quark.

Equation (1.3.26) makes no reference to deep inelastic scattering. It is a property of the proton alone, which is the root of universality: any process whose hard part reduces to a quark line leaving and re-entering the proton measures the same function.

Gauge invariance: the Wilson line

In QCD the product $\bar\psi(0)\psi(\lambda n)$ is not gauge invariant, because the two fields transform with different colour rotations. The handbag diagram alone is incomplete. The missing diagrams are those in which the struck quark exchanges any number of gluons with the remnant through the field component $n\cdot A$. They are not suppressed by powers of Q, and they sum to a path-ordered exponential of the gluon field $A^\mu = A^{a\mu}t^a$ along the line joining the two points:

$$\begin{aligned} q(x) &= \int \frac{d\lambda}{4\pi}\; e^{i\lambda x}\;\langle P|\,\bar\psi(0)\,\not n\,\mathcal{W}[0,\lambda n]\;\psi(\lambda n)\,|P\rangle, \\ \mathcal{W}[0,\lambda n] &= \mathcal{P}\exp\Big[-ig\int_0^{\lambda} d\sigma\; n\cdot A(\sigma n)\Big]. \end{aligned} \tag{1.3.27}$$

Here g is the strong coupling, with covariant derivative $D^\mu = \partial^\mu - igA^\mu$, and $\mathcal{P}$ orders the colour matrices along the path. $\mathcal{W}$ is a Wilson line: the colour phase the struck quark accumulates as it leaves through the field of the remnant.

Equation (1.3.27) is the modern definition of the quark distribution [4, 5]. In the light-cone gauge $n \cdot A = 0$ the Wilson line is the identity and Eq. (1.3.26) is recovered, which is why chapter 1.4 uses that gauge.

The number-density interpretation

Equation (1.3.27) becomes a count of quarks only under three conditions.

First, the Dirac structure. In the frame of Eq. (1.3.10), $\not{n} = (\gamma^0 + \gamma^3)/2P$, and so

$$\gamma^0 \not{n} = \frac{1}{P}\,\Pi_+, \qquad \Pi_+ = \tfrac{1}{2}\big(1 + \gamma^0\gamma^3\big), \qquad \bar\psi\, \not{n}\, \psi = \frac{1}{P}\,\big(\Pi_+\psi\big)^\dagger\big(\Pi_+\psi\big). \tag{1.3.28}$$

Π_+ is a Hermitian projector onto two of the four components of the quark field. For a quark moving along $+z$ at high energy these are the two large components, one for each helicity. So at $\lambda = 0$ the operator is a positive density, $|\Pi_+\psi|^2/P$.

Second, the gauge. With $\mathcal{W} = 1$, expanding $\Pi_+\psi$ in creation and annihilation operators for quarks of definite $n \cdot k$ turns Eq. (1.3.26) into the expectation value of the number of quarks with momentum fraction x, summed over transverse momentum. In any other gauge the operator contains arbitrarily many gluon fields.

Third, a fast-moving proton. ESW make the point with the lowest moment. Integrating Eq. (1.3.26) over x gives the local operator $\frac{1}{2}\bar\psi\not{n}\psi = (j^0 + j^3)/4P$, where $j^\mu = \bar\psi\gamma^\mu\psi$ is the quark number current. This is the number density $j^0 = \psi^\dagger\psi$ divided by $2P$ only when $j^3 \simeq j^0$, that is, when every quark moves along $+z$ at nearly the speed of light (ESW Eqs. 4.43 and 4.44). In the rest frame the same matrix element is a correlation over the distances of Eq. (1.3.9), not a snapshot.

Antiquarks and gluons

Equation (1.3.26) defines $q(x)$ for all real x. It vanishes for $|x| > 1$, because the states between the two fields have momentum fraction $1 - x \geq 0$. For negative x the field $\psi(\lambda n)$ creates an antiquark instead of removing a quark, and reordering the two anticommuting fields gives

$$q(-x) = -\,\bar q(x), \qquad 0 < x < 1, \tag{1.3.29}$$

where $\bar q$ is the antiquark distribution. One function on $-1 < x < 1$ holds both.

The gluon distribution is defined in the same way from the field strength $G^{\mu\nu}$, with a Wilson line in the adjoint representation between the two fields [4]:

$$x\, g(x) = -\int \frac{d\lambda}{2\pi}\, e^{i\lambda x}\; n_\mu n_\nu\, \langle P|\, G^{a\,\mu\alpha}(0)\, G^{a\,\nu}{}_{\alpha}(\lambda n)\, |P\rangle. \tag{1.3.30}$$

In light-cone gauge $n_\mu G^{\mu\alpha} = n \cdot \partial\, A^\alpha$ and the right-hand side counts transversely polarised gluons. The normalisation is fixed by the momentum sum rule of section 8.

The scale, and the definition in Moore's thesis

Equation (1.3.20) integrates over all transverse momenta, and section 6 showed that with point-like couplings the integral diverges logarithmically. This is an ultraviolet divergence of the operator in Eq. (1.3.27), removed like any other by renormalisation. The renormalised operator depends on a scale μ, and so does the distribution: $q(x) \to q(x, \mu^2)$. Roughly, $q(x, \mu^2)$ counts quarks with transverse momentum up to μ. The standard scheme is $\overline{\text{MS}}$.

Moore's thesis reaches the same object from the other side [2]. It keeps the parton-model formula, treats the distribution in it as a bare quantity $q_0(x,\epsilon)$ in $d = 4 - 2\epsilon$ dimensions, computes the collinear divergence of the one-gluon correction, and absorbs the pole into a redefinition (his Eq. 1.73, in the notation of this guide):

$$q(x,\mu^2) = q_0(x,\epsilon) - \frac{\alpha_S}{2\pi}\int_x^1 \frac{d\xi}{\xi}\, q_0(\xi,\epsilon)\, P_{qq}\Big(\frac{x}{\xi}\Big)\left[\frac{1}{\epsilon} - \gamma_E + \ln 4\pi - \ln\frac{\mu^2}{\mu_d^2}\right]. \tag{1.3.31}$$

Here P_{qq} is the splitting function of Eq. (1.4.12), γ_E is Euler's constant and μ_d is the mass unit of dimensional regularisation. This is Eq. (1.4.15) with the cut-off logarithm replaced by a pole.

The two constructions define the same function. The ultraviolet divergence of the operator and the collinear divergence of the parton-model cross-section are the same integral $\int dk_T^2/k_T^2$ read from its two ends: the operator must be cut off from above, the hard cross-section from below, and the subtraction at μ divides the integral between them. That the two agree to all orders is a result of Collins and Soper [4], quoted here without proof.

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8. Sum rules and positivity

The operator definition turns statements about the integral of a distribution into statements about local operators. Where the local operator is a conserved current, the integral is fixed exactly.

Moments are local operators

Multiply Eq. (1.3.27) by x^{N-1} and integrate over x. Each power of x becomes a derivative with respect to λ acting on the matrix element, the integral over x gives $\delta(\lambda)$, and the derivative of the Wilson line turns ∂ into the covariant derivative D (derivation D7):

$$\int_{-1}^{1} dx\; x^{N-1}\, q(x) = \frac{1}{2}\, n_{\mu_1}\cdots n_{\mu_N}\, \langle P|\, \bar\psi\, \gamma^{\mu_1}\, iD^{\mu_2}\cdots iD^{\mu_N}\, \psi\, |P\rangle. \tag{1.3.32}$$

The moments of chapter 1.4, Eq. (1.4.24), are matrix elements of a tower of local operators, and the anomalous dimensions $\gamma(N)$ are the anomalous dimensions of those operators. By Eq. (1.3.29) the left-hand side is $\int_0^1 dx\, x^{N-1}\big[q(x) + (-1)^N \bar q(x)\big]$.

The first moment: counting valence quarks

For $N = 1$ the operator is the quark number current $j^\mu = \bar\psi\gamma^\mu\psi$ of one flavour. Its charge counts quarks minus antiquarks of that flavour, which the strong interaction conserves. For a state normalised to $2E$ particles per unit volume, a conserved charge N_q gives $\langle P|j^\mu(0)|P\rangle = 2p^\mu N_q$. Then Eq. (1.3.32) reads $\frac{1}{2}\, n_\mu\, 2p^\mu N_q = N_q$:

$$\int_0^1 dx\, \big[q(x,\mu^2) - \bar q(x,\mu^2)\big] = N_q, \qquad N_u = 2, \quad N_d = 1, \quad N_s = 0. \tag{1.3.33}$$

The values are those of the proton, and $N_q = 0$ for the heavier flavours as well. These are the valence sum rules (ESW Eq. 4.24, Moore Eq. 1.92). They are exact: a conserved current is not renormalised, so they hold at every scale and every order, which in chapter 1.4 appears as $\gamma_{qq}(1) = 0$. They constrain differences only. The integral of $q + \bar q$ corresponds to no conserved current and is not fixed.

The second moment: momentum

For $N = 2$ the operator is $\bar{\psi}\gamma^{\mu} i D^{\nu} \psi$. Symmetrised in its indices, this is the quark part of the energy–momentum tensor, $T_q^{\mu\nu}$. The gluon moment from Eq. (1.3.30) gives the gluon part, $T_g^{\mu\nu} = -G^{a\,\mu\alpha} G^{a\,\nu}{}_{\alpha} + \frac{1}{4} g^{\mu\nu} G^2$, whose last term drops out because $n^2 = 0$. Write the forward matrix element of each part as $\langle P | T_i^{\mu\nu} | P \rangle = 2 A_i \, p^{\mu} p^{\nu}$ for a massless proton. Then (derivation D7)

$$\int_0^1 dx \; x \left[q(x, \mu^2) + \bar{q}(x, \mu^2) \right] = A_q(\mu^2), \qquad \int_0^1 dx \; x \, g(x, \mu^2) = A_g(\mu^2). \tag{1.3.34}$$

A_i is the fraction of the proton's momentum carried by parton species i. The total energy–momentum tensor is conserved and its charge is the momentum itself, so $\langle P | T^{\mu\nu} | P \rangle = 2 p^{\mu} p^{\nu}$ with no free coefficient. Hence $\sum_q A_q + A_g = 1$:

$$\int_0^1 dx \; x \left[\sum_q \left(q(x, \mu^2) + \bar{q}(x, \mu^2) \right) + g(x, \mu^2) \right] = 1. \tag{1.3.35}$$

This is the momentum sum rule (Moore Eq. 1.93). The sum is exact and independent of μ. The separate pieces are not: momentum flows between quarks and gluons as the scale changes, towards the division of Eq. (1.4.30).

The sum rule reveals the gluons. Measured structure functions fix the quark term, and it comes to about one half (ESW Eq. 4.24):

$$\sum_q \int_0^1 dx \; x \left[q(x) + \bar{q}(x) \right] \approx 0.5. \tag{1.3.36}$$

The other half is carried by constituents with no electric or weak charge, attributed to gluons.

Positivity

At lowest order a distribution is a number density, Eq. (1.3.28), and cannot be negative. Beyond lowest order the quantity that enters cross-sections is the renormalised distribution, Eq. (1.3.31): a bare density minus a divergent subtraction. Nothing in that definition forces it to be positive. What must be positive is every physical cross-section.

Whether $\overline{\text{MS}}$ distributions are nevertheless positive is not settled. Moore's thesis records the state of the question [2]: Candido, Forte and Hekhorn argued that they are; Collins, Rogers and Sato disputed an assumption of that argument, showing that the bare distributions need not be positive in $4 - 2\epsilon$ dimensions; the first group replied [6]. NNPDF imposes positivity in its fits, and our Phase B fits use NNPDF's full set of positivity and integrability constraints, recorded in the project ledger as 17 + 2.

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9. The proton in the infinite-momentum frame

This section describes the proton in the frame where the parton picture holds.

A contracted disc

Boost the proton to momentum $P \gg M$. Its internal motions are slowed by the time-dilation factor P/M, which is what freezes the partons during the collision, Eq. (1.3.1). Its longitudinal size is Lorentz-contracted by the same factor. A proton of radius R becomes a disc of radius R and thickness $2R\,M/P$. Partons with a finite fraction x of the momentum sit on this disc, with their transverse positions unchanged.

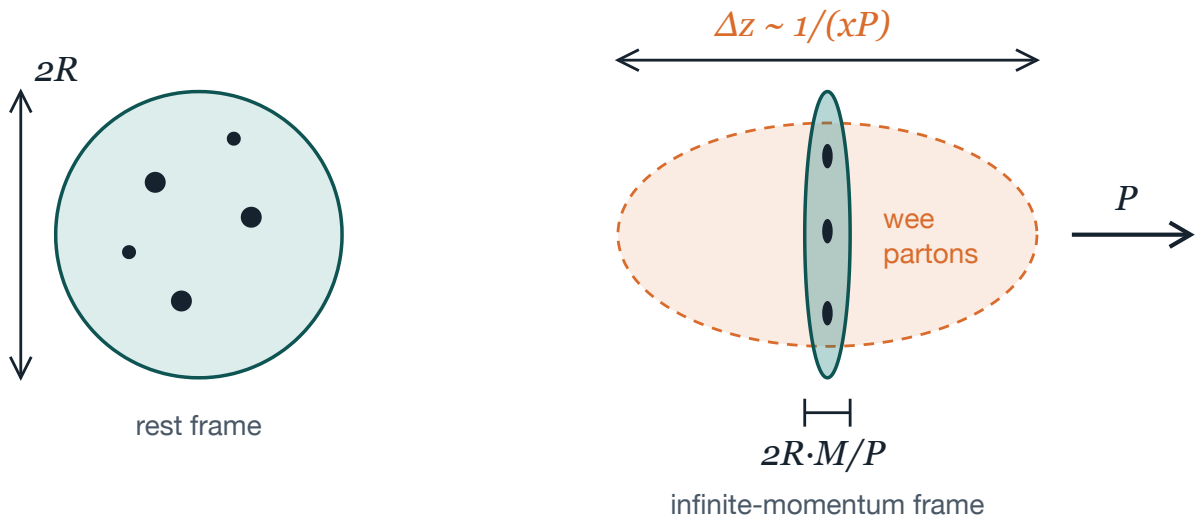

Figure 4. The proton at rest and at large momentum P. Partons with a finite momentum fraction lie on a disc of thickness $2RM/P$ (teal). Partons with very small x have too little longitudinal momentum to be localised that well, and spread over $\Delta z \sim 1/xP$ (orange). After ESW Fig. 4.5.

Wee partons

A parton with momentum fraction x has longitudinal momentum xP. By the uncertainty principle it cannot be localised along z more finely than

$$\Delta z \sim \frac{1}{xP}. \tag{1.3.37}$$

This is ESW's Eq. (4.45). It exceeds the thickness of the disc when

$$\frac{1}{xP} > \frac{2RM}{P} \quad \Longleftrightarrow \quad x < \frac{1}{2RM} \approx 0.13 \quad \text{for } R = 0.84 \text{ fm}. \tag{1.3.38}$$

The momentum P has dropped out. Partons with xP of the order of an ordinary hadronic momentum, about 1 GeV or less, are called wee partons [3]; they extend over about a fermi whatever the boost. At $P = 920$ GeV, the proton beam momentum at HERA, the disc is 0.0017 fm thick, while a parton at $x = 10^{-4}$ is spread over 2.1 fm.

Equation (1.3.38) is Eq. (1.3.9) in another frame: the rest-frame correlation length $1/Mx$, contracted by M/P, is $1/xP$. Both say that below $x \sim 0.1$ the probe is sensitive to distances larger than the proton.

The picture of independent partons on a thin disc is therefore least secure at small x, where partons overlap along z and can interact with one another (chapter 3.3).

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10. Factorisation, universality, and what the picture leaves out

What has been established

The handbag diagram with a damped blob gives a structure function that is a product of a hard part and a distribution. In QCD the product becomes a convolution, with corrections. At the level of a statement (chapter 1.6 treats it fully):

$$F_2(x,Q^2) = x \sum_{i=q,\bar{q},g} \int_x^1 \frac{d\xi}{\xi}\, C_i\Big(\frac{x}{\xi}, \frac{Q^2}{\mu^2}, \alpha_S(\mu^2)\Big)\, f_i(\xi,\mu^2) \;+\; O\Big(\frac{m^2}{Q^2}\Big). \tag{1.3.39}$$

The f_i are the distributions of Eqs. (1.3.27) and (1.3.30), renormalised at the factorisation scale μ. The coefficient functions C_i are calculable as power series in α_S; at lowest order $C_q = e_q^2\,\delta(1-x/\xi)$ and $C_g = 0$, which is the parton model. The remainder is a power correction, called higher twist: the corrections to Eq. (1.3.17), the F_L of Eq. (1.3.21), and diagrams where the photons attach to different quarks.

Because the f_i are defined by the proton alone, the same functions appear in every hard process with a proton in the initial state. For the production of a lepton pair in a proton–proton collision (Moore Eq. 1.85):

$$\sigma = \sum_{i,j} \hat{\sigma}_{ij} \otimes f_i \otimes f_j. \tag{1.3.40}$$

Here $\hat{\sigma}_{ij}$ is the cross-section for partons i and j and $\otimes$ the convolution of Eq. (1.4.19). Distributions measured in one set of processes predict another.

Neither equation is proved here. For deep inelastic scattering the proof uses the operator product expansion, the systematic form of the light-cone argument of section 3 [7]. For hadron–hadron processes the standard proofs are those of Collins, Soper and Sterman [5]; Moore's thesis, section 1.1.7, summarises the approaches [2].

What the picture leaves out

The derivation of scaling rested on the damping of $B(k,p)$ at large k_T^2. In QCD a quark acquires transverse momentum by radiating a gluon, with a probability that falls only as dk_T^2/k_T^2 up to the kinematic limit $k_T^2 \sim Q^2$. Three things follow, as in the model of section 6.

1. The integral in Eq. (1.3.20) diverges logarithmically. The distribution must be renormalised and acquires a scale, $q(x,\mu^2)$.
2. The approximation (1.3.17) fails for $k_T^2 \sim Q^2$. That region is a calculable correction to the hard scattering: the $O(\alpha_S)$ term of the coefficient function.
3. F_2 depends on $\ln Q^2$, and F_L is of order α_S instead of m^2/Q^2.

By Eq. (1.3.1), a fluctuation with $k_T^2 \sim Q^2$ is not frozen during the collision and cannot be assigned to the proton. Dividing the k_T integral into a part below μ that belongs to the distribution and a part above μ that belongs to the hard scattering is factorisation. Chapter 1.4 carries it out, starting from Eqs. (1.3.10), (1.3.12) and (1.3.13) with a single quark as the target.

In this project

- **What is parametrised.** The 52 free parameters of the MSHT20 form describe $\overline{\text{MS}}$ distributions, Eq. (1.3.27), at $\mu = Q_0 = 1.65$ GeV. The hard parts of Eq. (1.3.39) are taken at NNLO from NNPDF theory 40000000, with $\alpha_S(M_Z) = 0.118$.
- **Sum rules are imposed exactly.** The model fixes four further parameters analytically (ledger: "52 free parameters, 4 analytic sum rules"): the momentum sum rule, Eq. (1.3.35), and the number rules of Eq. (1.3.33) for u, d and s. Because they hold at every scale, imposing them at Q_0 is enough.
- **Positivity is a constraint, not a theorem.** It enters Phase B through the 17 + 2 conditions noted in section 8.

Derivations

D1. The hadronic tensor as a current commutator

Derives (1.3.4)

The Fourier representation. Start from Eq. (1.3.3) and write the delta function as $(2\pi)^4\delta^4(q+p-p_X) = \int d^4z\, e^{i(q+p-p_X)\cdot z}$.

Translation. The momentum operator $\hat{P}$ generates translations: $j(z) = e^{i\hat{P}\cdot z}\, j(0)\, e^{-i\hat{P}\cdot z}$. Acting on momentum eigenstates,

$$\langle P|\, j_\beta^\dagger(z)\,|X\rangle = e^{i(p-p_X)\cdot z}\,\langle P|\, j_\beta^\dagger(0)\,|X\rangle.$$

The phase $e^{i(p-p_X)\cdot z}$ in the Fourier representation is therefore absorbed by moving the first current from 0 to z:

$$W_{\alpha\beta} = \frac{1}{4\pi}\int d^4z\, e^{iq\cdot z}\sum_X \langle P|\, j_\beta^\dagger(z)\,|X\rangle\langle X|\, j_\alpha(0)\,|P\rangle.$$

Completeness. The states X are now unrestricted, so $\sum_X |X\rangle\langle X| = 1$. This gives the first form of Eq. (1.3.4).

The commutator. With the currents in the other order, $\langle P|\, j_\alpha(0)\, j_\beta^\dagger(z)\,|P\rangle$, the same steps give $(2\pi)^4\delta^4(q+p_X-p)$. The intermediate state would need momentum $p-q$ and invariant mass squared $(p-q)^2 = M^2 - 2\nu - Q^2 < M^2$. It has the baryon number of the proton, and the proton is the lightest such state. So this term vanishes and can be subtracted freely, which gives the commutator in Eq. (1.3.4).

D2. The region of z that dominates

Derives (1.3.6), (1.3.7), (1.3.8), (1.3.9)

Light-cone components. With $a^\pm = a^0 \pm a^3$, $a^0b^0 - a^3b^3 = \frac{1}{2}(a^+b^- + a^-b^+)$. So $a\cdot b = \frac{1}{2}(a^+b^- + a^-b^+) - a_T\cdot b_T$ and $a^2 = a^+a^- - a_T^2$.

The photon in the rest frame. With $p = (M,0,0,0)$, $\nu = p\cdot q = Mq^0$. With the photon along $+z$, $q_T = 0$ and $q^2 = (q^0)^2 - (q^3)^2 = -Q^2$, so $q^3 = \sqrt{(q^0)^2 + Q^2}$. Then $q^+ = q^0 + q^3$ and $q^+q^- = -Q^2$ exactly. Substituting $q^0 = Q^2/2Mx$:

$$q^- = q^0 - \sqrt{(q^0)^2+Q^2} = -\frac{2Mx}{1+\sqrt{1+4M^2x^2/Q^2}}.$$

For $Q^2 \gg M^2x^2$ this is $-Mx$, and $q^+ = Q^2/|q^-| \to Q^2/Mx = 2\nu/M$. This is Eq. (1.3.6). At $x = 0.1$ and $Q^2 = 10\ \text{GeV}^2$ the exact value is $q^- = -0.09374$ GeV against $-Mx = -0.09383$ GeV.

The oscillation argument. The integrand of Eq. (1.3.4) is $e^{\frac{i}{2}q^+z^-}\, e^{\frac{i}{2}q^-z^+}$ times the matrix element. An integral of $e^{\frac{i}{2}q^+z^-}$ against a function that is smooth over a range much longer than $2/q^+$ cancels between neighbouring half-periods. Contributions come from $|z^-| \lesssim 2/q^+$, and likewise from $|z^+| \lesssim 2/|q^-|$. This gives Eq. (1.3.7).

Causality. The commutator vanishes unless $z^2 = z^+z^- - z_T^2 \geq 0$. So $z^+z^- \geq z_T^2 \geq 0$, and with Eq. (1.3.7), $z^2 \leq z^+z^- \lesssim 4/(q^+|q^-|) = 4/Q^2$. This is Eq. (1.3.8). The non-smooth points of the commutator are on the light cone itself, inside this region.

Units. With $\hbar c = 0.1973$ GeV fm and $M = 0.938$ GeV, $1/M = 0.2103$ fm, which gives Eq. (1.3.9).

D3. The decomposition and the two projections

Derives (1.3.11), (1.3.13)

Coefficients. Write $k = a\,p + b\,n + k_T$ and require $n \cdot k_T = p \cdot k_T = 0$. Contracting with n: $n \cdot k = a\,(n \cdot p) + b\,n^2 = a$. Contracting with p: $p \cdot k = a\,p^2 + b\,(p \cdot n) = b$. In the frame of Eq. (1.3.10), a vector orthogonal to both p and n has $k^0 - k^3 = 0$ and $k^0 + k^3 = 0$, so only its x and y components survive and its Minkowski square is $-k_T^2$. Then $k^2 = 2ab\,(n \cdot p) - k_T^2$, which is Eq. (1.3.11).

The measure. $k^0 = aP + b/2P$ and $k^3 = aP - b/2P$, with Jacobian $|\partial(k^0, k^3)/\partial(a, b)| = 1$. So $d^4k = da\,db\,d^2k_T$. At fixed $a = \xi$ and k_T, $k^2 = 2\xi b - k_T^2$ gives $db = dk^2/2\xi$, the measure in Eq. (1.3.16).

The photon. In general $q = (n \cdot q)\,p + \nu\,n + q_T$. The replacement $n \to n + c\,p + c_T$, with c_T transverse and $c = c_T^2/2$, preserves $n^2 = 0$ and $n \cdot p = 1$, and is used to set $n \cdot q = 0$. Then $q^2 = -q_T^2$.

Contraction with $n_\alpha n_\beta$. In Eq. (1.3.5), $n_\alpha n_\beta\, g^{\alpha\beta} = n^2 = 0$ and $n \cdot q = 0$, so the F_1 term vanishes. In the second term $n \cdot (p + q/2x) = 1$. Hence $n_\alpha n_\beta W^{\alpha\beta} = F_2/\nu$.

Contraction with $p_\alpha p_\beta$. Now $p^2 = 0$ and $(p \cdot q)^2/q^2 = -\nu^2/Q^2$, so the first term gives $-(\nu^2/Q^2)F_1$. In the second, $p \cdot (p + q/2x) = \nu/2x$, giving $(\nu^2/4x^2)(F_2/\nu)$. Multiplying by $4x^2/\nu$ and using $Q^2 = 2\nu x$:

$$\frac{4x^2}{\nu}\,p_\alpha p_\beta W^{\alpha\beta} = -\frac{4x^2\nu}{Q^2}F_1 + F_2 = F_2 - 2xF_1.$$

D4. The handbag diagram and scaling

Derives (1.3.14), (1.3.17), (1.3.19), (1.3.20)

The matrix element. In the handbag approximation the final state is a quark of momentum l and spin s, plus a remnant X'. The current $j_\alpha = e_q \bar\psi \gamma_\alpha \psi$ acts once: ψ removes a quark from the proton and $\bar\psi$ creates the outgoing quark, which is treated as free:

$$\langle l, s; X' |\, j_\alpha(0)\,| P \rangle = e_q\,\bar u_s(l)\,\gamma_\alpha\,\langle X' |\,\psi(0)\,| P \rangle.$$

This replacement is the assumption of incoherence and freedom.

The outgoing quark. Its phase space and spin sum are

$$\int \frac{d^3l}{(2\pi)^3 2E_l} \sum_s u_s(l)\bar u_s(l) = \int \frac{d^4l}{(2\pi)^4}\; 2\pi\,\delta^+(l^2)\; l\!\!\!/$$

Assembling. Insert both into Eq. (1.3.3) and define $k = l - q$, the momentum the proton supplied. The delta function becomes $\delta^4(p - k - p_{X'})$:

$$W_{\alpha\beta} = \frac{e_q^2}{4\pi}\int \frac{d^4k}{(2\pi)^4}\; 2\pi\,\delta\big((k+q)^2\big)\;\big[\gamma_\beta(k\!\!\!/ + q\!\!\!/)\gamma_\alpha\big]_{ij} \times \sum_{X'} \langle P|\bar\psi_i(0)|X'\rangle\langle X'|\psi_j(0)|P\rangle\;(2\pi)^4\delta^4(p - k - p_{X'}).$$

The blob. The sum over X' is done as in D1. Writing the delta function as $\int d^4z\, e^{i(p-k-p_{X'})\cdot z}$, translating $\bar\psi$ to the point z and using completeness gives $\int d^4z\, e^{-ik\cdot z}\langle P|\bar\psi_i(z)\psi_j(0)|P\rangle$. Translating both fields by $-z$ and renaming $z\to -z$ turns this into $\int d^4z\, e^{ik\cdot z}\langle P|\bar\psi_i(0)\psi_j(z)|P\rangle$. With the factor $2\pi/4\pi=\frac12$ this is B_{ji} of Eq. (1.3.15), and the expression for W is Eq. (1.3.14). The order of α and β in the trace does not matter for the symmetric contractions used below.

The delta function. From Eqs. (1.3.16) and (1.3.12), $k\cdot q=\xi\,(p\cdot q)+k_T\cdot q_T$, where the other terms vanish because $n^2=0$, $n\cdot q_T=0$ and $p\cdot q_T=0$. The last product is between transverse four-vectors and equals minus the two-dimensional one. So $(k+q)^2=k^2+2\xi\nu-2\,q_T\cdot k_T-Q^2$. Setting it to zero,

$$\xi=x+\frac{2\,q_T\cdot k_T-k^2}{2\nu}=x\Big[1+\frac{2\,q_T\cdot k_T-k^2}{Q^2}\Big],$$

which agrees with Eq. (1.4.7). If B confines k_T^2 and $|k^2|$ to values of order m^2, the bracket is $1+O(m/Q)$, and the term linear in k_T averages to zero over its direction. Dropping the correction gives Eq. (1.3.17).

The result. With Eq. (1.3.18) and $n\cdot(k+q)=\xi$, $F_2=\nu\,n_\alpha n_\beta W^{\alpha\beta}=\nu\,e_q^2\int\frac{d^4k}{(2\pi)^4}\,2\xi\,\mathrm{Tr}[\not n B]\,\frac{1}{2\nu}\delta(\xi-x)$. The ν cancels and the delta function sets the explicit ξ to x, which is Eq. (1.3.19).

The free quark. For a target that is a quark of momentum p, the only state X' is the vacuum, $\langle 0|\psi(z)|p,s\rangle=u_s(p)e^{-ip\cdot z}$, and the spin average is $\frac12\sum_s u\bar u=\not p/2$. Equation (1.3.15) gives $B=\frac12(2\pi)^4\delta^4(k-p)\,\not p/2$. Then $\mathrm{Tr}[\not n\not p]=4\,n\cdot p=4$ and $q(x)=\delta(1-x)$.

D5. The longitudinal structure function and the Callan–Gross relation

Derives (1.3.21), (1.3.22)

The projection. From Eqs. (1.3.13) and (1.3.14), $F_L=\frac{4x^2}{\nu}e_q^2\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}[\not p\not l\not p B]\,\delta(l^2)$ with $l=k+q$. Since $p^2=0$, $\not p\not l\not p=2(p\cdot l)\not p$. From Eq. (1.3.16), $p\cdot k=(k^2+k_T^2)/2\xi$, so $p\cdot l=\nu+(k^2+k_T^2)/2\xi$.

The leading term. With Eq. (1.3.17), $2(p\cdot l)\cdot\frac{1}{2\nu}=1+O(m^2/\nu)$. Hence Eq. (1.3.21).

Why the integral is small. Parity and the spin average restrict B to the form $B=A_1\not p+A_2\not k$ plus terms with an even number of gamma matrices, which drop out of both traces. A_1 and A_2 are scalar functions of k^2 and $p\cdot k$. Then

$$\mathrm{Tr}\big[\not n B\big]=4\,(A_1+\xi A_2),\qquad \mathrm{Tr}\big[\not p B\big]=4A_2\,(p\cdot k)=\frac{2A_2\,(k^2+k_T^2)}{\xi}.$$

The first is of order one. The second is proportional to $k^2+k_T^2$, which the damping holds at order m^2. So the integral in Eq. (1.3.21) is of order m^2 times that in Eq. (1.3.20), and $F_L/F_2\sim m^2/Q^2$. For the free quark of D4, $\mathrm{Tr}[\not p\not p]=4p^2=0$ and F_L vanishes identically.

Spin-0 constituents. For a charged scalar the current vertex is $(2k+q)^\alpha$ and the blob is a scalar function $D(k,p)$, so $W^{\alpha\beta}\propto\int d^4k\,(2k+q)^\alpha(2k+q)^\beta D\,\delta((k+q)^2)$. The contractions are $n\cdot(2k+q)=2\xi$ and $p\cdot(2k+q)=\nu+2p\cdot k\simeq\nu$. Then $F_2\propto\nu\,(2x)^2\,\frac{1}{2\nu}=2x^2$ and $F_L\propto\frac{4x^2}{\nu}\,\nu^2\,\frac{1}{2\nu}=2x^2$, with the same constant of proportionality. So $F_L=F_2$ and $F_1=0$.

D6. The spectator model

Derives (1.3.23), (1.3.24), (1.3.25)

The blob. The remnant is one scalar of momentum $r = p - k$ and mass M_s. The amplitude for the proton, treated as a massless spin-½ particle, to emit it and leave an off-shell quark is $\langle r|\psi(0)|P\rangle = g\,\frac{\not k}{k^2}\,u(p)$. The sum over X' in D4 is the scalar's phase space, $\int \frac{d^3r}{(2\pi)^3 2E_r}(2\pi)^4\delta^4(p-k-r) = 2\pi\,\delta\big((p-k)^2 - M_s^2\big)$. With the spin average $\frac{1}{2}\sum u\bar u = \not p/2$ and the factor $\frac{1}{2}$ of Eq. (1.3.15):

$$B = \frac{1}{2}\cdot 2\pi\,\delta\big((p-k)^2 - M_s^2\big)\cdot|g|^2\,\frac{\not k\not p\not k}{2\,(k^2)^2},$$

which is Eq. (1.3.23).

Kinematics. From Eq. (1.3.16), $(p-k)^2 = k^2 - 2p\cdot k = -\big[(1-\xi)k^2 + k_T^2\big]/\xi$. Setting this to M_s^2 gives $k^2 = -(k_T^2 + \xi M_s^2)/(1-\xi)$, and $\delta\big((p-k)^2 - M_s^2\big) = \frac{\xi}{1-\xi}\,\delta\big(k^2 + \frac{k_T^2+\xi M_s^2}{1-\xi}\big)$. Also $2p\cdot k = k^2 - M_s^2$.

The traces. $\not k\not p\not k = 2(p\cdot k)\not k - k^2\not p$. So

$$\mathrm{Tr}\big[\not n\not k\not p\not k\big] = 4\big[2(p\cdot k)\,\xi - k^2\big] = 4\big[-(1-\xi)k^2 - \xi M_s^2\big] = 4k_T^2, \qquad \mathrm{Tr}\big[\not p\not k\not p\not k\big] = 8\,(p\cdot k)^2.$$

The distribution. In Eq. (1.3.20) use the measure (1.3.16); the delta functions do the ξ and k^2 integrals:

$$q(x) = \frac{1}{(2\pi)^4}\cdot\frac{1}{2x}\cdot\frac{x}{1-x}\cdot\frac{\pi}{2}\int d^2k_T\,\frac{4k_T^2\,|g|^2}{(k^2)^2} = \frac{1-x}{16\pi^2}\int dk_T^2\,\frac{k_T^2\,|g|^2}{(k_T^2 + xM_s^2)^2},$$

using $d^2k_T = \pi\,dk_T^2$ and $(k^2)^2 = (k_T^2 + xM_s^2)^2/(1-x)^2$. This is Eq. (1.3.24).

The exact structure function. Without the approximation (1.3.17), keep $\delta((k+q)^2) = \delta(A - B_\theta\cos\theta)$ with $A = k^2 + 2\xi\nu - Q^2$, $B_\theta = 2Qk_T$ and θ the angle between k_T and q_T. As in chapter 1.4, derivation D1, the angular integral gives $2/\sqrt{B_\theta^2 - A^2}$, and $B_\theta^2 - A^2 = 4\nu^2(\xi_+ - \xi)(\xi - \xi_-)$ is a quadratic in ξ at fixed virtuality $\kappa = |k^2|$. The result is

$$F_2(x,Q^2) = \frac{e_q^2}{16\pi^3}\int_{\kappa_0}^{2\nu - M_s^2} d\kappa\,\frac{|g(\kappa)|^2}{\kappa^2}\int_{\xi_-}^{\xi_+}\frac{d\xi\;\;\xi\,\big[(1-\xi)\kappa - \xi M_s^2\big]}{\sqrt{(\xi_+ - \xi)(\xi - \xi_-)}}, \qquad \kappa_0 = \frac{xM_s^2}{1-x}.$$

As $\nu\to\infty$ at fixed κ, $\xi_\pm \to x$ and the inner integral tends to $\pi x\,[(1-x)\kappa - xM_s^2]$, which returns $e_q^2\,x\,q(x)$. For constant $|g|^2 = g_0^2$ the integrand at large κ is $x(1-x)\,d\kappa/\kappa$ up to $\kappa\sim 2\nu$, which gives the logarithm in Eq. (1.3.25).

The longitudinal part. Replacing the F_2 trace by the F_L one, with $2\,p\cdot l = 2\nu - \kappa - M_s^2$ and $8(p\cdot k)^2 = 2(\kappa + M_s^2)^2$:

$$F_L(x,Q^2) = \frac{e_q^2}{16\pi^2}\,\frac{x^2}{\nu^2}\int_{\kappa_0}^{2\nu - M_s^2} d\kappa\,|g(\kappa)|^2\,\frac{(2\nu - \kappa - M_s^2)\,(\kappa + M_s^2)^2}{\kappa^2}.$$

For the damped vertex the integral grows as 2ν times a convergent integral, so $F_L \propto 1/\nu$. For constant $|g|^2$ it grows as $\int^{2\nu} d\kappa\,(2\nu - \kappa) = 2\nu^2$, so $F_L \to e_q^2\,(g_0^2/16\pi^2)\,2x^2$.

The check script evaluates both integrals and confirms the limits and the numbers quoted in section 6.

D7. The light-cone operator, its moments and the sum rules

Derives (1.3.26), (1.3.32), (1.3.33), (1.3.34), (1.3.35)

From the blob to the operator. Put Eq. (1.3.15) into Eq. (1.3.20):

$$q(x) = \frac{1}{2}\int d^4z \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot z}\, \delta(n\cdot k - x)\, \langle P|\, \bar\psi(0)\, \not{n}\, \psi(z)\, |P\rangle .$$

Decompose both vectors as in Eq. (1.3.11): $k = \xi p + b\, n + k_T$ and $z = (n\cdot z)\, p + \lambda\, n + z_T$ with $\lambda = p\cdot z$. Then $k\cdot z = \xi\lambda + b\,(n\cdot z) - k_T\cdot z_T$, and $d^4k = d\xi\, db\, d^2k_T$, $d^4z = d\lambda\, d(n\cdot z)\, d^2z_T$. The matrix element does not depend on k, so the integrals over b and k_T are those of pure phases:

$$\int db\; e^{ib\,(n\cdot z)} = 2\pi\,\delta(n\cdot z), \qquad \int d^2k_T\; e^{-ik_T\cdot z_T} = (2\pi)^2\,\delta^2(z_T).$$

These set $z = \lambda n$. The ξ integral sets $\xi = x$. Collecting, $\tfrac{1}{2}\cdot(2\pi)^3/(2\pi)^4 = 1/4\pi$, which gives Eq. (1.3.26).

Moments. Let $f(\lambda) = \langle P|\bar\psi(0)\not{n}\,\mathcal{W}[0,\lambda n]\,\psi(\lambda n)|P\rangle$. Because $q(x)$ vanishes for $|x|>1$, the integral over x can be extended to the whole line, and $x^{N-1}e^{i\lambda x} = (-i\partial_\lambda)^{N-1}e^{i\lambda x}$:

$$\int dx\; x^{N-1} q(x) = \int \frac{d\lambda}{4\pi}\, f(\lambda)\,(-i\partial_\lambda)^{N-1}\, 2\pi\,\delta(\lambda) = \frac{1}{2}\,(i\partial_\lambda)^{N-1} f(\lambda)\Big|_{\lambda=0},$$

after integrating by parts $N-1$ times.

The covariant derivative. From Eq. (1.3.27), $\partial_\lambda\big(\mathcal{W}[0,\lambda n]\,\psi(\lambda n)\big) = \mathcal{W}\,\big(n\cdot\partial - ig\, n\cdot A\big)\psi = \mathcal{W}\, n\cdot D\,\psi$. Each derivative brings down one $n\cdot D$, and at $\lambda = 0$ the Wilson line is the identity. With $\not{n} = n_\mu\gamma^\mu$ this is Eq. (1.3.32).

Normalisation of a conserved charge. Let $\hat N = \int d^3x\, j^0(x)$ with $\hat N|P\rangle = N_q|P\rangle$. With $\langle P'|P\rangle = 2E\,(2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})$, translating the current gives $\langle P'|\hat N|P\rangle = (2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})\,\langle P|j^0(0)|P\rangle$. Comparing, $\langle P|j^0(0)|P\rangle = 2E\,N_q$, and by covariance $\langle P|j^\mu(0)|P\rangle = 2p^\mu N_q$. With $N=1$ in Eq. (1.3.32): $\int_{-1}^{1} q\,dx = \tfrac{1}{2}\,n_\mu\,2p^\mu N_q = N_q$. Splitting the range with Eq. (1.3.29), $\int_{-1}^{0} q(x)\,dx = -\int_0^1 \bar q(x)\,dx$, which gives Eq. (1.3.33).

Momentum. The same argument with $\hat P^\nu = \int d^3x\, T^{0\nu}$ gives $\langle P|T^{\mu\nu}(0)|P\rangle = 2p^\mu p^\nu$ for the total tensor. The quark part is $T_q^{\mu\nu} = \frac{i}{4}\bar\psi\big(\gamma^\mu \overleftrightarrow{D}^\nu + \gamma^\nu \overleftrightarrow{D}^\mu\big)\psi$. In a forward matrix element total derivatives vanish, so $n_\mu n_\nu\langle T_q^{\mu\nu}\rangle = n_\mu n_\nu\langle\bar\psi\gamma^\mu iD^\nu\psi\rangle$, and Eq. (1.3.32) with $N=2$ is $\tfrac{1}{2}\cdot 2A_q\,(n\cdot p)^2 = A_q$. On the left, Eq. (1.3.29) gives $\int_{-1}^{0} x\,q(x)\,dx = \int_0^1 x\,\bar q(x)\,dx$.

Gluons. Integrating Eq. (1.3.30) over all x gives $-n_\mu n_\nu\langle G^{a\,\mu\alpha}G^{a\,\nu}{}_{\alpha}(0)\rangle = n_\mu n_\nu\langle T_g^{\mu\nu}\rangle = 2A_g$. The function $x\,g(x)$ is even in x, so the integral over $0<x<1$ is A_g. Adding the quark flavours, $\sum_q A_q + A_g = 1$, which is Eq. (1.3.35).

The check script tests Eq. (1.3.26) and the first three moments on a trial distribution.

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