1.2 The parton model and Bjorken scaling

Point-like constituents, the Callan–Gross relation, and what the neutral- and charged-current data say about flavour.

1. What this chapter establishes

Chapter 1.1 reduced deep inelastic scattering (DIS) to two functions, $F_1(x, Q^2)$ and $F_2(x, Q^2)$, which hold everything a virtual photon can learn about the proton. That chapter made no statement about what the functions look like. This chapter does. It has one experimental input and one model.

The input is Bjorken scaling. At large Q^2 the measured structure functions depend on the dimensionless variable x and hardly at all on Q^2. The model is the parton model: the photon scatters elastically off a single point-like constituent that is free during the collision. The chapter derives the following results from it.

1. Scaling is what point-like constituents give, and an extended charge distribution gives something very different (sections 2 and 3).
2. The constituents can be treated as free because a fast-moving proton is frozen on the time scale of the collision (section 4).
3. The kinematic variable x is the fraction of the proton's momentum carried by the struck parton, and $F_2(x) = x \sum_q e_q^2 \, q(x)$ (section 5).
4. The relation $F_2 = 2xF_1$ holds if and only if the charged partons have spin ½. Spin-0 partons would give $F_1 = 0$ (section 6).
5. The distributions $q(x)$ obey exact sum rules. Measured against them, quarks carry only about half of the proton's momentum (section 7).
6. Proton, neutron and neutrino data together separate the flavours, test the fractional quark charges, and show that the sea is not flavour symmetric (sections 8 to 10).

The chapter follows the line of Ellis, Stirling and Webber (ESW), section 4.1 [1], with the tensor version of the same calculation in Moore's thesis, section 1.1.2 [2]. The model is heuristic. Chapter 1.3 rebuilds it from field theory, and chapter 1.4 computes the corrections that QCD makes to it. Every formula stated here has been checked numerically; the script is linked at the end.

2. The cross-section and the observation of scaling

What is carried over from chapter 1.1

An electron of momentum k scatters off a proton of momentum P and leaves with momentum k'. The exchanged photon has momentum $q = k - k'$, which is space-like. The invariants are

$$Q^2 = -q^2, \qquad \nu = P \cdot q, \qquad x = \frac{Q^2}{2\nu}, \qquad y = \frac{q \cdot P}{k \cdot P}, \qquad s = (k + P)^2.$$

Q^2 measures the resolving power of the photon, y is the fraction of the electron's energy transferred to the proton in the proton's rest frame, and s is the squared centre-of-mass energy. With the proton mass M neglected against Q, they are related by $Q^2 = x\,y\,s$. Both x and y lie between 0 and 1.

In the same limit the cross-section of chapter 1.1 is

$$\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[y^2\,F_1(x,Q^2) + (1-y)\,\frac{F_2(x,Q^2)}{x}\right]. \tag{1.2.1}$$

Here α is the fine-structure constant. The factor $1/Q^4$ is the photon propagator squared. The two structure functions enter with different functions of y. At fixed x and Q^2, $y = Q^2/xs$ can be changed only by changing the beam energy, and that is how experiments separate F_1 from F_2.

Equation (1.2.1) is equivalent to the form used by ESW [1], which groups the terms as $[1+(1-y)^2]F_1 + (1-y)(F_2 - 2xF_1)/x$. The two brackets are identical because $1+(1-y)^2 = y^2 + 2(1-y)$.

The observation

The Bjorken limit is $Q^2 \to \infty$ and $\nu \to \infty$ with x fixed. Bjorken proposed in 1969 [3] that in this limit the structure functions become functions of x alone:

$$F_i(x,Q^2) \;\longrightarrow\; F_i(x), \qquad i = 1, 2. \tag{1.2.2}$$

The SLAC–MIT experiments of 1968–69 [4] scattered electrons of up to about 20 GeV off hydrogen and measured the inelastic cross-section at large momentum transfer for the first time. Two things were found. The inelastic cross-section at large Q^2 was far larger than the rapid fall of the elastic cross-section had led people to expect. And at fixed x the structure function F_2 was independent of Q^2 within the errors.

Later experiments extended the range. ESW's Figure 4.2 [1] collects F_2 for a proton target from SLAC–MIT, from the muon experiment BCDMS and from the electron–proton collider experiments H1 and ZEUS. The values of Q^2 run from 1.5 to 1000 GeV², nearly three orders of magnitude, and the points lie close to a single curve in x. Scaling is approximate, and section 11 returns to the departures from it. The first task is to understand why it holds at all.

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3. Why scaling means point-like constituents

Scaling is a statement about dimensions. F_2 is dimensionless. If the photon scatters off an object that has a size $1/Q_0$, then F_2 can depend on the dimensionless ratio Q/Q_0, and it will. A structure function that does not depend on Q^2 means the photon meets no length scale in the object it strikes. Elastic scattering shows what the alternative looks like.

An extended charge

Take a static charge distribution $\rho(\vec{r})$, normalised to unit total charge. The amplitude for an electron to scatter off it with momentum transfer $\vec{q}$ is the point-charge amplitude multiplied by the Fourier transform of the density. This is the form factor:

$$F(\vec{q}) = \int d^3r\;\rho(\vec{r})\,e^{i\vec{q}\cdot\vec{r}}, \qquad \frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} \big|F(\vec{q})\big|^2. \tag{1.2.3}$$

For a distribution of radius R, the phase $\vec{q}\cdot\vec{r}$ varies by about $|\vec{q}|R$ across the charge. When $|\vec{q}|R \ll 1$ the whole charge scatters coherently and $F \approx 1$. When $|\vec{q}|R \gg 1$ the contributions from different parts of the distribution cancel and $F \to 0$. Only a point charge, $\rho = \delta^3(\vec{r})$, has $F = 1$ at every momentum transfer.

The proton's measured elastic form factors follow the dipole form closely:

$$G_D(Q^2) = \left(1 + \frac{Q^2}{0.71\ \text{GeV}^2}\right)^{-2}, \qquad \langle r^2 \rangle = -6\,\frac{dG_D}{dQ^2}\bigg|_{Q^2=0} = \frac{12}{0.71\ \text{GeV}^2} = 0.66\ \text{fm}^2. \tag{1.2.4}$$

The dipole is the Fourier transform of an exponential charge density. The root-mean-square radius is 0.81 fm. At $Q^2 = 20$ GeV² the form factor is $G_D = 1.2 \times 10^{-3}$, and the elastic rate is suppressed relative to a point charge by $G_D^2 = 1.4 \times 10^{-6}$.

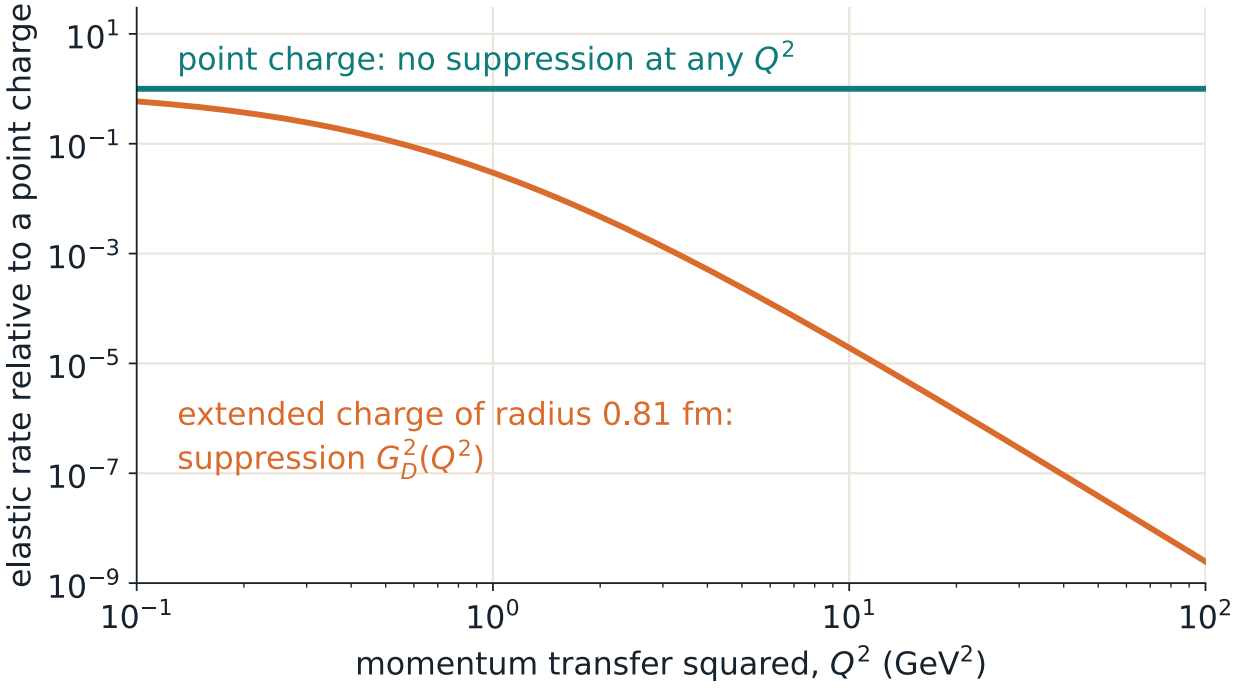

Figure 1. What an extended charge does and a point charge does not. The orange curve is $G_D^2(Q^2)$ from Eq. (1.2.4), the factor by which the proton's finite size suppresses elastic scattering. It falls by more than eight orders of magnitude between $Q^2 = 0.1$ and 100 GeV². The teal line is a point charge. The inelastic structure function F_2 at fixed x behaves like the teal line.

The inference

The proton as a whole is an extended object, and the elastic channel shows it: the chance that the proton absorbs a hard photon and stays intact dies quickly with Q^2. The inelastic channel does not die. At fixed x it does not change at all, to a first approximation. So whatever absorbs the photon in an inelastic collision has no measurable size: it is point-like down to the shortest distance probed, about $1/Q$.

If the constituents themselves had a radius $1/Q_0$, each would carry its own form factor, F_2 would fall as a function of Q/Q_0 once Q exceeded Q_0, and scaling would fail there. Scaling up to a given Q is therefore an upper limit on the size of the constituents. Feynman named the constituents *partons* [5]. The next two sections turn the inference into a calculation.

4. The frozen proton

The calculation will treat the struck parton as a free particle. That needs justifying, because partons inside a proton interact strongly and all the time. The justification is a separation of time scales, and it is clearest in a frame in which the proton moves fast.

Two clocks

Let the proton have momentum P along the z axis, with $P \gg M$. Its energy is $\sqrt{P^2 + M^2} \approx P + M^2/2P$. Suppose that at some instant it consists of partons $i = 1, 2, \ldots$, each with a fraction ξ_i of the longitudinal momentum, a transverse momentum k_{Ti} and a mass m_i, with $\sum_i \xi_i = 1$. Expanding each parton's energy in the same way, the energy of this configuration exceeds the proton's by (derivation D2)

$$\Delta E = \frac{\Delta}{2P}, \qquad \Delta = \sum_i \frac{k_{Ti}^2 + m_i^2}{\xi_i} - M^2. \tag{1.2.5}$$

Δ is built from hadronic scales, a few hundred MeV, and does not depend on P. By the uncertainty principle the configuration lasts for a time $\tau_{\text{fluct}} \sim 1/\Delta E = 2P/\Delta$. This is time dilation: the faster the proton moves, the more slowly its internal state changes.

The photon sets the second clock. In the frame of chapter 1.4, Eq. (1.4.3), the photon is $q = \nu\, n + q_T$ and its energy is $q^0 = \nu/2P = Q^2/4xP$. The absorption takes a time of order $\tau_{\text{int}} \sim 1/q^0$. The ratio of the two times is

$$\frac{\tau_{\text{int}}}{\tau_{\text{fluct}}} \sim \frac{4xP}{Q^2} \cdot \frac{\Delta}{2P} = \frac{2x\,\Delta}{Q^2}. \tag{1.2.6}$$

The momentum P has cancelled. The ratio is a hadronic scale squared divided by Q^2, and it vanishes in the Bjorken limit. For three partons with $\xi = (0.5, 0.3, 0.2)$, masses of 0.3 GeV and transverse momenta of 0.3 GeV on two of them, $\Delta = 0.53$ GeV². At $x = 0.25$ and $Q^2 = 20$ GeV² the ratio is 0.013.

What follows

Three assumptions of the parton model rest on this ratio being small.

- **The impulse approximation.** During the absorption the partons do not have time to interact with one another. The photon meets a parton that is free for the duration of the collision.
- **Incoherence.** The photon is absorbed by one parton, and the cross-sections for the different partons add as probabilities. Interference between scattering off two different partons would need the struck parton to exchange momentum of order Q with another during the short time τ_{int}, and is suppressed by powers of $1/Q^2$.
- **Collinear partons.** The transverse momenta and masses of the partons are of the size of $\sqrt{\Delta}$ and are neglected against Q. A parton then has momentum ξP, parallel to the proton's.

What happens after the collision does not matter for the inclusive cross-section. The struck parton and the remnants turn into hadrons on the slow clock, with probability one, and the sum over all final states X cannot change the rate.

5. Electron–quark scattering and the parton model

The elementary process

The hard process is elastic scattering of the electron off a massless point-like quark of charge e_q (in units of the positron charge e):

$$e^-(k) + q(p_q) \;\to\; e^-(k') + q(p_q'), \qquad p_q = \xi P.$$

It is the same process as $e^-\mu^- \to e^-\mu^-$ with a different charge. One photon is exchanged, and the amplitude is

$$\mathcal{M} = \frac{e^2 e_q}{\hat{t}}\,\big[\bar{u}(k')\gamma^\mu u(k)\big]\big[\bar{u}(p_q')\gamma_\mu u(p_q)\big].$$

Use the Mandelstam variables of the electron–quark system, marked with hats:

$$\hat{s} = (k + p_q)^2, \qquad \hat{t} = (k - k')^2, \qquad \hat{u} = (p_q - k')^2, \qquad \hat{s} + \hat{t} + \hat{u} = 0.$$

Squaring the amplitude, averaging over the spins of the incoming particles and summing over those of the outgoing ones gives (derivation D1)

$$\overline{\sum}|\mathcal{M}|^2 = 2\,e_q^2\,e^4\,\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^{\,2}}. \tag{1.2.7}$$

The photon does not change the quark's colour, so the colour average gives a factor of one.

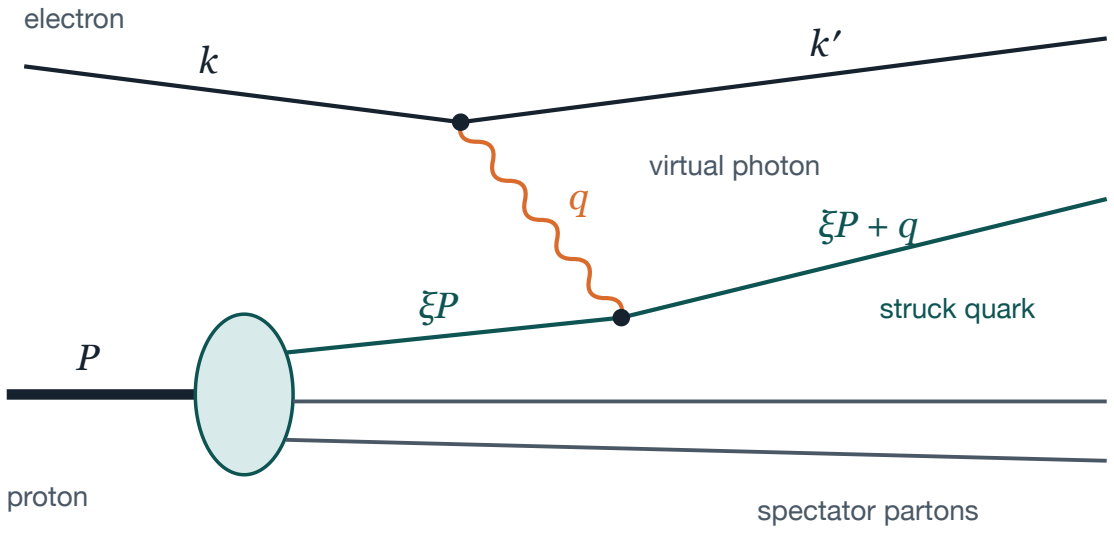

Figure 2. Deep inelastic scattering in the parton model. In a frame where the proton moves fast, it is a set of partons moving parallel to it. The photon is absorbed by one of them, which carries a fraction ξ of the proton's momentum. The others take no part. The requirement that the struck quark is a real massless particle after the collision, $(\xi P + q)^2 = 0$, forces $\xi = x$.

From invariants to DIS variables

The three partonic invariants are fixed by quantities measured on the electron. With all masses neglected,

$$\hat{t} = -Q^2, \qquad \hat{s} = 2\,k\cdot p_q = \xi\, s, \qquad \hat{u} = -2\,p_q\cdot k' = -\hat{s}\,(1-y). \tag{1.2.8}$$

The last relation uses $p_q\cdot k' = p_q\cdot k - p_q\cdot q$ and $y = q\cdot P/k\cdot P = q\cdot p_q/k\cdot p_q$. The fraction ξ cancels in y, so y is the same for the electron–proton and the electron–quark system.

For massless two-body scattering the cross-section is $d\hat{\sigma}/d\hat{t} = \overline{\sum}|\mathcal{M}|^2/16\pi\hat{s}^2$ (derivation D1). Insert Eqs. (1.2.7) and (1.2.8) and $e^2 = 4\pi\alpha$:

$$\frac{d\hat{\sigma}}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}\left[1 + (1-y)^2\right]. \tag{1.2.9}$$

The hat marks a cross-section for a quark target. Two features matter later. The cross-section has no scale in it other than Q^2 itself. And its dependence on y is the fixed function $1 + (1-y)^2$, which section 6 traces to the quark's spin.

Bjorken x is the parton's momentum fraction

The quark leaves the collision with momentum $p_q' = \xi P + q$. It is a real massless particle, so

$$p_q'^{\,2} = (\xi P + q)^2 = 2\,\xi\, P\cdot q - Q^2 = 2\nu\,(\xi - x) = 0 \qquad \Longrightarrow \qquad \xi = x. \tag{1.2.10}$$

This is the central identification of the model. The variable x was defined in chapter 1.1 from the electron's momenta alone. Equation (1.2.10) says that it equals a property of the proton's interior: the fraction of the proton's momentum carried by the parton that absorbed the photon. An experiment at a given x selects partons of that momentum fraction.

Written as a distribution in x, the quark cross-section is Eq. (1.2.9) with a delta function that enforces Eq. (1.2.10):

$$\frac{d^2\hat{\sigma}}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[1 + (1-y)^2\right]\,\frac{1}{2}\,e_q^2\,\delta(x-\xi). \tag{1.2.11}$$

Comparing with Eq. (1.2.1), term by term in y (derivation D3), gives the structure functions of a single quark of momentum ξP:

$$\hat{F}_2(x) = x\,e_q^2\,\delta(x-\xi) = 2x\,\hat{F}_1(x). \tag{1.2.12}$$

The incoherent sum

A real proton does not give a delta function. The measured F_2 is a smooth distribution in x, so the partons carry a range of momentum fractions. Define $q(\xi)\,d\xi$ as the number of quarks of flavour q in the proton with momentum fraction between ξ and $\xi + d\xi$, and $\bar{q}(\xi)$ likewise for antiquarks. These are the parton distribution functions (PDFs). By section 4 the photon scatters off the partons incoherently, so the proton's structure functions are the quark's, weighted by these densities:

$$F_2(x) = 2x\,F_1(x) = \sum_{q,\bar{q}} \int_0^1 d\xi\; q(\xi)\,x\,e_q^2\,\delta(x-\xi) = x\sum_{q,\bar{q}} e_q^2\,q(x). \tag{1.2.13}$$

The parton model in one line. $F_2(x) = x\sum_q e_q^2\,[q(x)+\bar q(x)]$. The right side does not contain Q^2: scaling is derived, and it follows from the absence of any length scale in point-like scattering. F_2/x counts the quarks and antiquarks at momentum fraction x, each weighted by its squared charge. F_2 itself is their momentum density with the same weights.

Moore's thesis reaches Eq. (1.2.13) by the tensor route [2]: the proton state in the hadronic tensor is replaced by a free quark of momentum ξP weighted by $f_q(\xi)$, the phase space of the outgoing quark supplies $\delta\big((\xi P+q)^2\big) = \delta(x-\xi)/2\nu$, and projecting out F_1 and F_2 gives the same result.

Where it is approximate

Equation (1.2.10) dropped the proton mass and the parton's transverse momentum. Keeping the mass, the struck parton has $p_q^2 = \xi^2 M^2$ and the condition becomes $\xi^2 M^2 + 2\xi\nu - Q^2 = 0$, with the solution

$$\xi = \frac{2x}{1+\sqrt{1+4x^2M^2/Q^2}} = x\left[1 - \frac{x^2M^2}{Q^2} + \dots\right]. \tag{1.2.14}$$

At $x = 0.3$ and $Q^2 = 4$ GeV² the correction is 2%. These *target-mass corrections*, and the effects of transverse momentum and of interference between partons, all fall as powers of $1/Q^2$. They are one reason fits discard data at low Q^2 and low invariant mass of the hadronic final state. The corrections that fall only as logarithms are the subject of chapter 1.4.

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6. The Callan–Gross relation and the spin of the partons

The relation

Equation (1.2.13) contains two statements. One is the formula for F_2. The other is a relation between the two structure functions, due to Callan and Gross [6]:

$$F_2(x) = 2x\,F_1(x), \qquad \text{equivalently} \qquad F_L \equiv F_2 - 2xF_1 = 0. \tag{1.2.15}$$

F_L is the longitudinal structure function. In the ESW grouping of the cross-section, $[1+(1-y)^2]F_1 + (1-y)F_L/x$, the first term is the absorption of transversely polarised virtual photons and the second the absorption of longitudinally polarised ones [1]. The Callan–Gross relation says that the partons do not absorb longitudinal photons.

Why spin ½ gives it

The cleanest argument uses the frame in which the photon carries no energy and the struck quark reverses its direction: it comes in with momentum $+\frac{1}{2}Q$ along the collision axis and leaves with $-\frac{1}{2}Q$. The coupling $\bar u\gamma^\mu u$ of a massless fermion to a vector field conserves helicity, the projection of the spin on the direction of motion. The direction of motion reverses, so the spin must reverse too. The component of the quark's angular momentum along the axis therefore changes by one unit. A transverse photon has helicity ±1 and can supply it. A longitudinal photon has helicity 0 and cannot. Hence $F_L = 0$.

What scalar partons would give

Repeat the calculation of section 5 for a charged parton of spin 0. The photon vertex is now $-ie\,e_q\,(p_q + p_q')^\mu$, and the result (derivation D3) is

$$\left.\frac{d\hat{\sigma}}{dQ^2}\right|_{\text{spin }0} = \frac{4\pi\alpha^2 e_q^2}{Q^4}\,(1-y) \qquad \Longrightarrow \qquad F_1 = 0, \quad F_L = F_2 = x\sum e_q^2\,\phi(x), \tag{1.2.16}$$

where $\phi(x)$ is the density of scalar partons. The helicity argument explains this as well. A spin-0 particle has no spin to flip, so its angular momentum along the axis cannot change, and it absorbs only longitudinal photons. The two cases are opposite extremes:

CHARGED PARTONS	FACTOR IN THE CROSS-SECTION	TRANSVERSE FUNCTION	LONGITUDINAL FUNCTION
spin ½	$1+(1-y)^2$	$F_1 = F_2/2x$	$F_L = 0$
spin 0	$2(1-y)$	$F_1 = 0$	$F_L = F_2$

Measurements find $F_L \ll F_2$ [1]. The charged constituents of the proton have spin ½. Together with the charges found in section 9, this identifies them with the quarks of hadron spectroscopy. F_L is not exactly zero. In QCD it starts at order α_S [1], because a quark that has radiated a gluon carries transverse momentum and the helicity argument no longer forbids longitudinal photons.

The y dependence as a helicity statement

The function $1+(1-y)^2$ can be read in the same way. In the centre-of-mass frame of the electron and the quark, let θ be the scattering angle of the electron. Then (derivation D4)

$$1 - y = -\frac{\hat{u}}{\hat{s}} = \cos^2\frac{\theta}{2}, \qquad \hat{s}^2 + \hat{u}^2 = \hat{s}^2\Big[\underbrace{1}_{\text{equal helicities}} + \underbrace{(1-y)^2}_{\text{opposite helicities}}\Big]. \tag{1.2.17}$$

Small y is forward scattering and $y = 1$ is backward scattering. The electromagnetic interaction conserves the helicity of the electron and of the quark separately, so there are four helicity combinations, in two classes.

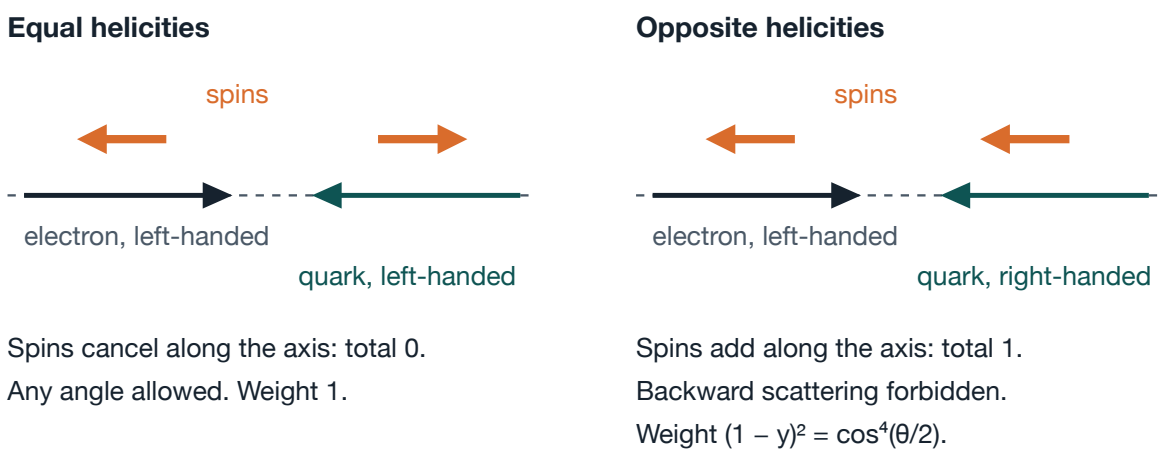

Figure 3. The two helicity classes in the electron–quark centre-of-mass frame. Black and teal arrows are the momenta, orange arrows the spins. A left-handed particle has its spin opposite to its momentum. Left: with equal helicities the spin projections on the collision axis cancel, the state has no preferred direction, and the scattering is isotropic. Right: with opposite helicities the projections add to one unit. Scattering through 180° would reverse both spins and change the projection by two units, which is forbidden, so the distribution vanishes at $y = 1$.

A state with one unit of angular momentum along the axis scatters with the amplitude $\frac{1}{2}(1+\cos\theta) = 1 - y$, which is the rotation function $d^1_{11}(\theta)$ of angular momentum theory. This reading becomes a tool in section 9. Neutrinos are purely left-handed, so in neutrino scattering the two terms of Eq. (1.2.17) are no longer added with equal weight. One of them is selected by whether the target is a quark or an antiquark.

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7. Parton distributions and their sum rules

Flavours, valence and sea

With the quark charges $e_u = e_c = +\frac{2}{3}$ and $e_d = e_s = -\frac{1}{3}$, Eq. (1.2.13) for a proton target reads, for four flavours [1],

$$F_2^p(x) = x\left[\frac{4}{9}\left(u + \bar{u} + c + \bar{c}\right) + \frac{1}{9}\left(d + \bar{d} + s + \bar{s}\right)\right]. \tag{1.2.18}$$

Every distribution here is a function of x and refers to the proton. The photon sees each flavour only in the combination $q + \bar{q}$, because the charge enters squared.

The proton has the quantum numbers of two up quarks and one down quark. It also contains quark–antiquark pairs, which in QCD come from gluons splitting. The pairs add equal numbers of quarks and antiquarks of a flavour, so the difference $q - \bar{q}$ is not affected by them. This suggests the decomposition

$$q_v(x) \equiv q(x) - \bar{q}(x), \qquad q(x) = q_v(x) + q_{\text{sea}}(x), \qquad q_{\text{sea}}(x) = \bar{q}(x). \tag{1.2.19}$$

The *valence* distributions u_v and d_v carry the proton's quantum numbers. The *sea* is everything else. The equality $q_{\text{sea}} = \bar{q}$ at each x is a convention for dividing q into two parts. Only its integral over x is forced.

Number sum rules

The integral of $q - \bar{q}$ over all momentum fractions is the net number of quarks of that flavour in the proton. These numbers are fixed by the proton's charge, baryon number and strangeness (derivation D5):

$$\int_0^1 dx\; u_v(x) = 2, \qquad \int_0^1 dx\; d_v(x) = 1, \qquad \int_0^1 dx\; \bigl[s(x) - \bar{s}(x)\bigr] = 0. \tag{1.2.20}$$

They are exact. Chapter 1.4 shows that QCD evolution preserves them, because the first moment of the quark splitting function vanishes. The integrals of q and $\bar{q}$ separately do not exist: the number of sea quarks grows without limit as $x \to 0$. Only the difference is finite.

The momentum sum rule

The partons together carry all of the proton's momentum. A parton at ξ carries a fraction ξ, so

$$\sum_q \int_0^1 dx\; x\,\bigl[q(x) + \bar{q}(x)\bigr] + \int_0^1 dx\; x\,g(x) = 1. \tag{1.2.21}$$

The last term allows for partons that the photon does not see. If quarks and antiquarks were the only constituents, the first term alone would equal one. That can be tested with data, because F_2 is a momentum density.

Integrate Eq. (1.2.18) and its neutron counterpart, Eq. (1.2.23) below, over x. Write $\varepsilon_u = \int_0^1 x(u + \bar{u})\,dx$ for the momentum fraction carried by up quarks and antiquarks in the proton, ε_d likewise, and neglect the small strange and charm terms:

$$\int_0^1 F_2^p\,dx = \frac{4}{9}\,\varepsilon_u + \frac{1}{9}\,\varepsilon_d, \qquad \int_0^1 F_2^n\,dx = \frac{1}{9}\,\varepsilon_u + \frac{4}{9}\,\varepsilon_d. \tag{1.2.22}$$

The measured integrals quoted in the textbook of Halzen and Martin [7] are 0.18 and 0.12. Solving the two equations gives $\varepsilon_u = 0.36$ and $\varepsilon_d = 0.18$. Up quarks carry twice the momentum of down quarks, as the valence count suggests. But the total is

$$\varepsilon_u + \varepsilon_d = 0.54.$$

ESW quote the same finding as $\sum_q \int x(q + \bar{q})\,dx \approx 0.5$ [1]. Quarks and antiquarks carry about half of the proton's momentum. The other half belongs to constituents with no electric charge and no weak charge. They are identified with the gluons of QCD, and $g(x)$ in Eq. (1.2.21) is their distribution. This was the first evidence for gluons as constituents of the proton. Gluons are seen directly in other processes: jets and photons at large transverse momentum, and the production of heavy quarks [1]. In DIS they show up through the Q^2 dependence of F_2 (chapter 1.4).

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8. Flavour from proton and neutron targets

A single structure function gives one combination of distributions at each x. Separating flavours needs several targets and several probes. This section uses the neutron. The next uses neutrinos.

Isospin

The strong interaction is very nearly symmetric under the exchange of up and down quarks, and that exchange turns a proton into a neutron. So the up quarks in a neutron are distributed as the down quarks in a proton, and vice versa: $u^n = d^p \equiv d$, $d^n = u^p \equiv u$, and the same for antiquarks. The strange and charm distributions are the same in both. All distributions can then be kept as proton distributions, and

$$F_2^n(x) = x\left[\frac{4}{9}\left(d + \bar{d} + c + \bar{c}\right) + \frac{1}{9}\left(u + \bar{u} + s + \bar{s}\right)\right]. \tag{1.2.23}$$

There is no free neutron target. F_2^n is obtained from deuterium, which to a first approximation is a loosely bound proton and neutron, $F_2^d \approx \frac{1}{2}(F_2^p + F_2^n)$ per nucleon. The corrections for nuclear binding and for the motion of the nucleons grow at large x and limit how well the neutron is known there.

The ratio and its bounds

Write $U = u + \bar{u}$, $D = d + \bar{d}$ and $S = s + \bar{s}$, and drop charm. Each is a number density and cannot be negative. The ratio of neutron to proton structure functions is then confined to a fixed interval at every x (derivation D6):

$$\frac{F_2^n}{F_2^p} = \frac{U + 4D + S}{4U + D + S}, \qquad \frac{1}{4} \;\le\; \frac{F_2^n}{F_2^p} \;\le\; 4. \tag{1.2.24}$$

The lower bound is reached only where the proton contains nothing but up quarks, and the upper bound only where it contains nothing but down quarks. The bounds depend on the squared charges $\frac{4}{9}$ and $\frac{1}{9}$ and on nothing else, so they test the charge assignment.

The measured ratio respects the bounds. It is close to 1 at small x and falls steadily as x grows, towards the lower end of the allowed interval. Each limit is informative.

- **Small x, ratio near 1.** The ratio is 1 where $U = D$. The valence quarks cannot give that, since there are twice as many u as d. A sea of pairs that is nearly the same for up and down, and much larger than the valence at small x, gives it.
- **Large x, ratio well below $\frac{2}{3}$.** With no sea and $d_v = \frac{1}{2}u_v$, the value suggested by counting quarks, the ratio would be $(1 + 2)/(4 + \frac{1}{2}) = \frac{2}{3}$. The data fall below this. So d_v/u_v decreases as $x \to 1$: a quark carrying most of the proton's momentum is much more likely to be an up quark.

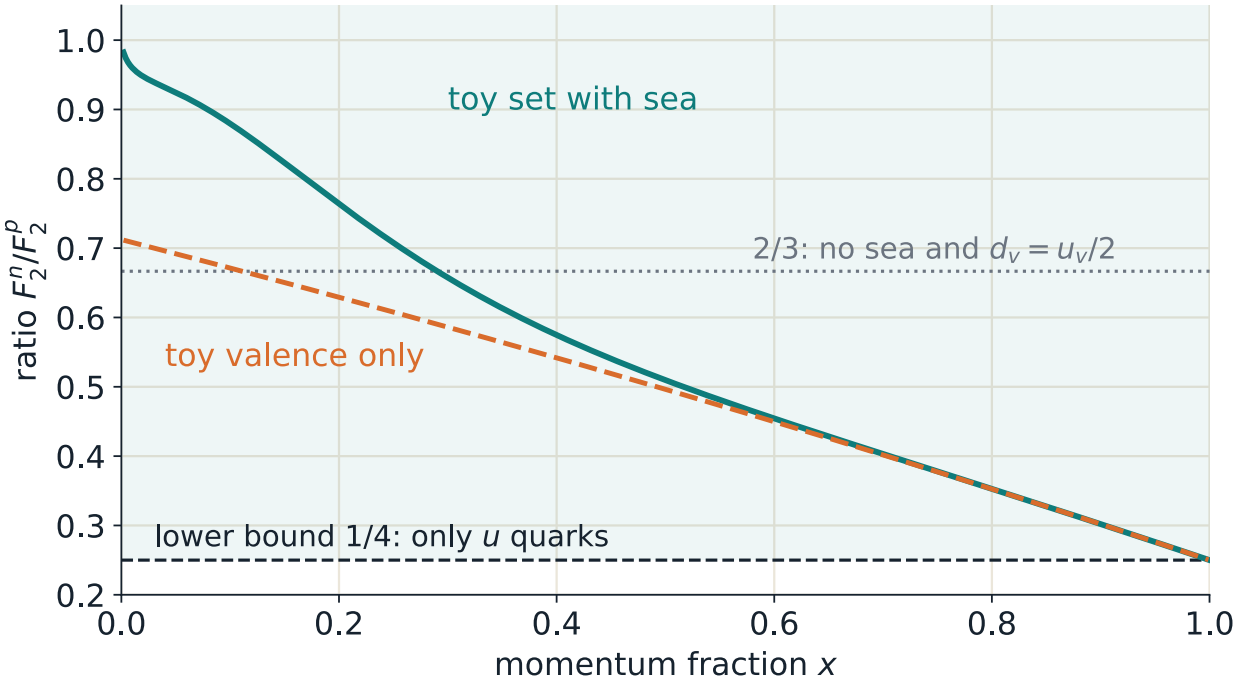

Figure 4. The ratio F_2^n/F_2^p computed from the illustrative distributions of Figure 5 (these are not data). Solid: the full set. Dashed orange: the valence quarks alone. The sea pulls the ratio to 1 at small x. At large x the ratio approaches the bound $\frac{1}{4}$ of Eq. (1.2.24) because the toy d_v falls faster than u_v by one power of $(1-x)$. The dotted line at $\frac{2}{3}$ is the value for $d_v = u_v/2$ with no sea.

The Gottfried sum rule

The difference of the two structure functions isolates the difference between up and down. From Eqs. (1.2.18) and (1.2.23), with $u + \bar{u} = u_v + 2\bar{u}$ and the same for d,

$$F_2^p - F_2^n = \frac{x}{3}\left[(u+\bar{u}) - (d+\bar{d})\right] = \frac{x}{3}\left[u_v - d_v\right] + \frac{2x}{3}\left[\bar{u} - \bar{d}\right]. \tag{1.2.25}$$

Strange and charm cancel. Divide by x, integrate, and use the number sum rules (1.2.20):

$$S_G \equiv \int_0^1 \frac{dx}{x}\left[F_2^p - F_2^n\right] = \frac{1}{3} + \frac{2}{3}\int_0^1 dx\,\left[\bar{u}(x) - \bar{d}(x)\right]. \tag{1.2.26}$$

If the sea had equal numbers of $\bar{u}$ and $\bar{d}$, the integral would be exactly $\frac{1}{3}$. This is the Gottfried sum rule [8]. Unlike Eq. (1.2.20), it is an assumption about the sea, and no conservation law protects it.

The New Muon Collaboration measured $S_G = 0.235 \pm 0.026$ from muon scattering on hydrogen and deuterium [9]. By Eq. (1.2.26),

$$\int_0^1 dx\,\left[\bar{d}(x) - \bar{u}(x)\right] = \frac{3}{2}\left(\frac{1}{3} - S_G\right) = 0.148 \pm 0.039.$$

The proton's sea contains more $\bar{d}$ than $\bar{u}$. The Drell–Yan process, in which an antiquark from one hadron annihilates a quark from another, later confirmed the excess and measured its dependence on x [10]. Perturbative gluon splitting produces $u\bar{u}$ and $d\bar{d}$ pairs at the same rate, so the asymmetry has a non-perturbative origin. ESW record

that it was not quantitatively understood [1]. For a fit, the consequence is practical: the parametrisation must give $\bar{u}$ and $\bar{d}$ separate freedom.

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9. Flavour from neutrinos

Charged-current neutrino scattering, $\nu_\mu N \to \mu^- X$ and $\bar{\nu}_\mu N \to \mu^+ X$, probes the proton with a W boson. It differs from photon exchange in three ways that make it a flavour filter. The W changes the flavour of the quark it strikes. It couples with the same strength to every quark, with no factors of $\frac{4}{9}$ and $\frac{1}{9}$. And it couples only to left-handed particles and right-handed antiparticles.

The elementary cross-sections

A W^+, emitted when $\nu_\mu \to \mu^-$, can be absorbed by d, s, $\bar{u}$ or $\bar{c}$. Take $Q^2 \ll M_W^2$, so that the W propagator reduces to the Fermi constant G_F, set the quark-mixing matrix to the identity and neglect the charm mass. The neutrino is left-handed, so only one helicity class of Figure 3 occurs for each target (derivation D4):

$$\frac{d\hat{\sigma}}{dy}\big(\nu\, q \to \mu^- q'\big) = \frac{G_F^2\,\hat{s}}{\pi}, \qquad \frac{d\hat{\sigma}}{dy}\big(\nu\, \bar{q} \to \mu^- \bar{q}'\big) = \frac{G_F^2\,\hat{s}}{\pi}\,(1-y)^2. \tag{1.2.27}$$

A left-handed quark meeting a left-handed neutrino is the equal-helicity case and scatters isotropically. An antiquark is right-handed, which is the opposite-helicity case, with the factor $(1-y)^2$. For antineutrinos the roles are exchanged: $\bar{\nu}\,\bar{q}$ is isotropic and $\bar{\nu}\,q$ carries $(1-y)^2$.

Weighting with the parton densities and using $\hat{s} = xs$ gives the proton cross-sections

$$\begin{aligned} \frac{d^2\sigma^{\nu p}}{dx\,dy} &= \frac{G_F^2\,s}{\pi}\,x\Big[\,d + s + (1-y)^2\,(\bar{u}+\bar{c})\Big], \\ \frac{d^2\sigma^{\bar{\nu} p}}{dx\,dy} &= \frac{G_F^2\,s}{\pi}\,x\Big[\,\bar{d} + \bar{s} + (1-y)^2\,(u+c)\Big]. \end{aligned} \tag{1.2.28}$$

The y distribution separates quarks from antiquarks. The flat part of the neutrino cross-section measures quarks and the part falling as $(1-y)^2$ measures antiquarks. Electromagnetic scattering cannot do this, since both helicities of the electron contribute and the photon sees $q + \bar{q}$.

Structure functions

Because the weak interaction violates parity, the general neutrino cross-section has a third structure function. In the massless limit of ESW's Eq. (4.3) [1],

$$\frac{d^2\sigma^{\nu(\bar{\nu})}}{dx\,dy} = \frac{G_F^2\,s}{2\pi}\Big[(1-y)\,F_2 + y^2\,xF_1 \;\pm\; y\big(1-\tfrac{1}{2}y\big)\,xF_3\Big],$$

with the upper sign for neutrinos. Comparing with Eq. (1.2.28) gives, for a proton target [1],

$$\begin{aligned} F_2^{\nu} &= 2x\,\big[d + s + \bar{u} + \bar{c}\big], & xF_3^{\nu} &= 2x\,\big[d + s - \bar{u} - \bar{c}\big], \\ F_2^{\bar{\nu}} &= 2x\,\big[u + c + \bar{d} + \bar{s}\big], & xF_3^{\bar{\nu}} &= 2x\,\big[u + c - \bar{d} - \bar{s}\big], \end{aligned} \tag{1.2.29}$$

together with $2xF_1 = F_2$ in each case. F_2 is the sum of the quark and antiquark terms and xF_3 is their difference. The factor 2 relative to Eq. (1.2.13) is the weak charge: the W couples through $\gamma^\mu(1-\gamma_5)$, which is twice the left-handed

projection.

Neutrino experiments use heavy targets with nearly equal numbers of protons and neutrons. For such an *isoscalar* target N, average each structure function over proton and neutron using isospin. With $s = \bar{s}$ and $c = \bar{c}$, and averaging xF_3 over neutrino and antineutrino beams,

$$F_2^{\nu N} = x \sum_q \left[q + \bar{q}\right] \equiv x\,\Sigma(x), \qquad \tfrac{1}{2}\left(xF_3^{\nu N} + xF_3^{\bar{\nu} N}\right) = x\left[u_v + d_v\right]. \tag{1.2.30}$$

These are the two cleanest combinations in the subject. $F_2^{\nu N}$ is the momentum density of all quarks and antiquarks with unit weights. xF_3 is the valence distribution, with the sea removed.

Three tests

The number of valence quarks. Integrating the second relation of Eq. (1.2.30) with the number sum rules gives the Gross–Llewellyn Smith sum rule [11]:

$$\int_0^1 dx\ \tfrac{1}{2}\left[F_3^{\nu N}(x) + F_3^{\bar{\nu} N}(x)\right] = \int_0^1 dx\left[u_v + d_v\right] = 3. \tag{1.2.31}$$

It counts the valence quarks in a nucleon. Measurements at Q^2 of a few GeV² give a value near 2.5. The shortfall is accounted for by QCD, which multiplies the right side by $1 - \alpha_S/\pi$ at leading order.

The quark charges. For the isoscalar target the electromagnetic structure function is the average of Eqs. (1.2.18) and (1.2.23). Dividing by $F_2^{\nu N}$ of Eq. (1.2.30) (derivation D6):

$$\frac{F_2^{eN}}{F_2^{\nu N}} = \frac{5}{18}\left[1 - \frac{3}{5}\,\frac{s + \bar{s} - c - \bar{c}}{\Sigma}\right], \qquad \frac{5}{18} = \frac{1}{2}\left(\frac{4}{9} + \frac{1}{9}\right). \tag{1.2.32}$$

The number $\frac{5}{18}$ is the mean squared charge of the up and down quarks. Two different probes, one electromagnetic and one weak, measure the same distributions, and the ratio of the results is fixed by the quark charges. The data agree with $\frac{5}{18}$ once the small strange correction is included. This is direct evidence that the partons carry the fractional charges $\frac{2}{3}$ and $-\frac{1}{3}$, and that the partons seen by the photon and by the W are the same objects. For the illustrative set of section 10, strange quarks carry 7% of the quarks' momentum and the ratio of the integrated structure functions is 0.266, against $\frac{5}{18} = 0.278$.

The antiquark content. Integrate Eq. (1.2.28) over y, using $\int_0^1 (1-y)^2 dy = \frac{1}{3}$, and over x, for an isoscalar target. Let $Q = \int_0^1 x(u+d)\,dx$ and $\bar{Q} = \int_0^1 x(\bar{u}+\bar{d})\,dx$ be the momentum fractions of light quarks and antiquarks, and neglect strange and charm. Then

$$R \equiv \frac{\sigma^{\bar{\nu}N}}{\sigma^{\nu N}} = \frac{\bar{Q} + \frac{1}{3}Q}{Q + \frac{1}{3}\bar{Q}}, \qquad \text{so} \qquad \frac{\bar{Q}}{Q} = \frac{3R - 1}{3 - R}. \tag{1.2.33}$$

A nucleon with no antiquarks would have $R = \frac{1}{3}$. The measured ratio is close to $\frac{1}{2}$, which gives $\bar{Q}/Q \approx 0.2$. Antiquarks are present and carry a modest share of the momentum.

Strange quarks

The strange distribution enters F_2^{eN} and $F_2^{\nu N}$ only as a small correction to much larger terms. A direct handle is the production of charm. A neutrino can turn a strange quark into a charm quark, $\nu_\mu\, s \to \mu^- c$. The competing transition from a down quark, $\nu_\mu\, d \to \mu^- c$, is suppressed by the small quark-mixing angle between the first and second

generations. The charm quark forms a hadron that can decay to a muon, so the signature is a pair of muons of opposite charge. The rate of such *dimuon* events measures $s(x)$ with neutrinos and $\bar{s}(x)$ with antineutrinos. The result, reflected in the distributions ESW show, is that the strange sea is roughly half the size of the average of the $\bar{u}$ and $\bar{d}$ seas at a scale of a few GeV [1].

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10. The picture that emerges

The naive expectation

Before the data are examined in detail, the simplest guess consistent with sections 7 to 9 is three valence quarks plus a sea that is the same for every light flavour [1]:

$$u = u_v + S, \qquad d = d_v + S, \qquad \bar{u} = \bar{d} = s = \bar{s} = S. \tag{1.2.34}$$

With the sum rules (1.2.20) and (1.2.21) this leaves four functions: u_v, d_v, S and g.

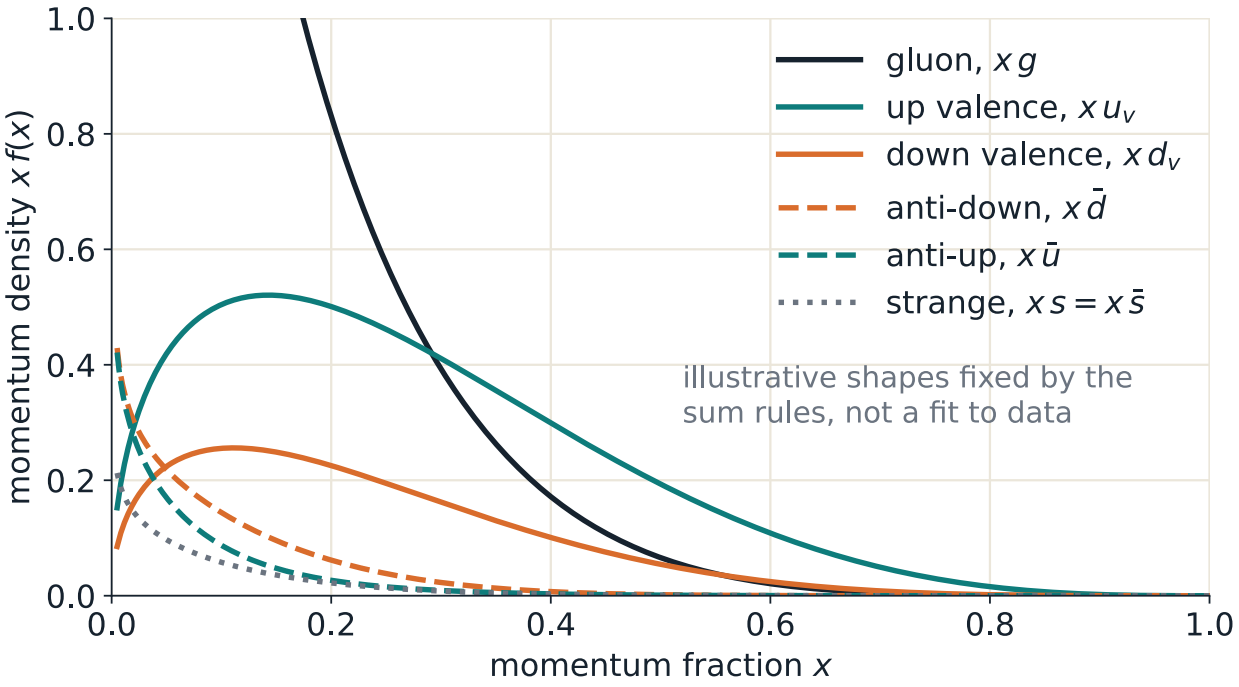

Figure 5. The typical shapes, drawn with illustrative functions $xf = A\,x^a(1-x)^c$ whose normalisations are fixed by the sum rules. This is not a fit. The set has $\int u_v = 2$, $\int d_v = 1$, $\int(\bar{d} - \bar{u}) = 0.148$, a strange sea half the light-sea average, and momentum fractions of 0.31 for the valence quarks, 0.17 for the sea and 0.52 for the gluon. Compare ESW Figure 4.3 [1], which shows a real fit at $\mu^2 = 10$ GeV².

What a fit shows

ESW's Figure 4.3 shows the distributions from a global fit of the time at $\mu^2 = 10$ GeV² [1]. Their general form is the one in Figure 5.

- **Valence.** $x\,u_v$ and $x\,d_v$ are humps that vanish at both ends, with maxima between $x = 0.1$ and 0.2. They are well below the value $x = \frac{1}{3}$ that three quarks sharing the momentum equally would give, because half the momentum

is in gluons and because the quarks radiate. u_v is about twice d_v near the peak and falls more slowly at large x, as section 8 required.

- **Sea.** The antiquark distributions are concentrated at small x. They rise as $x \to 0$ and are negligible above $x \approx 0.3$.
- **Gluon.** $x\,g$ is the largest distribution at small x by a wide margin and falls quickly with x. Above $x \approx 0.2$ the valence quarks dominate.

The fit departs from Eq. (1.2.34) in three ways.

- **The sea is not SU(3) symmetric.** The strange distribution is roughly a factor of two below the average of $\bar{u}$ and $\bar{d}$ (section 9).
- **The light sea is not isospin symmetric.** $\bar{d}$ exceeds $\bar{u}$ (section 8).
- **There is some charm.** At $\mu^2 = 10$ GeV² a small charm distribution is present, as expected when the photon can resolve $c\bar{c}$ pairs, $Q^2 \gtrsim m_c^2$.

ESW note that neither of the first two features was quantitatively understood. Both are non-perturbative properties of the proton that must be measured.

In this project

These sections explain the structure of our own fits and one of their main results.

- **The parametrisation.** Our fits use the MSHT20 form with 52 free parameters and four sum rules imposed analytically. The sum rules remove normalisations that the data would otherwise have to determine. The form gives $\bar{u}$, $\bar{d}$ and the strange distributions separate freedom, which the findings of sections 8 and 9 require.
- **What photon data cannot see.** The first Phase A fits used 648 points of F_2 on proton and deuteron targets from SLAC and BCDMS (33 and 34 points from SLAC, 333 and 248 from BCDMS). By Eqs. (1.2.18) and (1.2.23), such data fix two combinations of quark distributions at each x: $U - D$ from the proton–deuteron difference and $\frac{5}{18}(U + D) + \frac{1}{9}S$ from the deuteron. They cannot separate quarks from antiquarks, cannot isolate the strange distribution, and reach the gluon only indirectly.
- **Flat directions.** With the full set of DIS data, which adds neutrino measurements among others, 12 of the 52 parameter directions are flat in our fits, and 9 of 52 when Drell–Yan and W, Z data are added (Phase B ledger). The flattest directions are dominated by the strange parameters (Results, PPDF-20). This is the difficulty of section 9 in quantitative form: strangeness enters most observables as a small correction, and only dedicated channels such as dimuon production reach it directly.

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11. Scaling violations

The parton model predicts exact scaling. The data of section 2 scale only approximately, and the departure is systematic [1]:

- at large x, F_2 decreases slowly as Q^2 increases;
- at small x, F_2 increases with Q^2.

The change is logarithmic in Q^2. It is far too slow to be a form factor of the kind in Eq. (1.2.4), which would bring a power of Q^2, so it does not undo the conclusion that the partons are point-like. It also does not fall away as $Q^2 \to \infty$, so it is not one of the power corrections of section 5.

The parton model has no room for such an effect. Its partons are free, and Eq. (1.2.9) contains no scale. The missing ingredient is the interaction between the partons. A quark can radiate a gluon before it absorbs the photon, and a gluon can split into a quark–antiquark pair. A photon of larger Q^2 resolves more of this radiation. It finds the momentum of a fast quark shared with the gluons it has emitted, which depletes large x, and it finds more pairs from gluon splitting, which fills small x.

Two further chapters make this precise. Chapter 1.3 puts the parton model on a field-theory footing: the distributions become matrix elements of quark and gluon operators in the proton, and the sum rules of section 7 become statements about conserved currents. Chapter 1.4 computes the radiation of one gluon, finds the logarithm of Q^2, and derives the evolution equations that predict how every distribution changes with scale. The identification of x with a momentum fraction, Eq. (1.2.13) as the leading term of a systematic expansion, the sum rules and the flavour relations all survive.

Derivations

D1. Elastic electron–quark scattering

Derives (1.2.7), (1.2.9)

The amplitude squared. The amplitude in section 5 is a product of an electron current and a quark current. Square it, sum over all four spins and divide by 4 for the average over the two incoming spins. With massless spinors, $\sum_{\text{spins}} u(k)\bar{u}(k) = \not{k}$, each current gives a trace:

$$\overline{\sum}|\mathcal{M}|^2 = \frac{e^4 e_q^2}{\hat{t}^{\,2}} \cdot \frac{1}{4}\, L^{\mu\nu}\, H_{\mu\nu}, \qquad L^{\mu\nu} = \mathrm{Tr}\big[\not{k}'\gamma^\mu \not{k}\gamma^\nu\big], \quad H_{\mu\nu} = \mathrm{Tr}\big[\not{p}'_q\gamma_\mu \not{p}_q\gamma_\nu\big].$$

The traces. The trace of four gamma matrices is $\mathrm{Tr}[\not{a}\gamma^\mu\not{b}\gamma^\nu] = 4\,(a^\mu b^\nu + a^\nu b^\mu - g^{\mu\nu}\, a\cdot b)$. So

$$L^{\mu\nu} = 4\left(k'^\mu k^\nu + k'^\nu k^\mu - g^{\mu\nu}\, k\cdot k'\right), \qquad H_{\mu\nu} = 4\left(p'_{q\mu}p_{q\nu} + p'_{q\nu}p_{q\mu} - g_{\mu\nu}\, p_q\cdot p'_q\right).$$

The contraction. Multiply out. The terms without $g^{\mu\nu}$ give $2(k\cdot p_q)(k'\cdot p'_q) + 2(k\cdot p'_q)(k'\cdot p_q)$. The two cross terms with one metric tensor each give $-2(k\cdot k')(p_q\cdot p'_q)$, and the term with two gives $+4(k\cdot k')(p_q\cdot p'_q)$, since $g^{\mu\nu}g_{\mu\nu} = 4$. The terms in $(k\cdot k')(p_q\cdot p'_q)$ cancel:

$$L^{\mu\nu}H_{\mu\nu} = 32\left[(k\cdot p_q)(k'\cdot p'_q) + (k\cdot p'_q)(k'\cdot p_q)\right].$$

Invariants. For massless particles $\hat{s} = 2k\cdot p_q = 2k'\cdot p'_q$ and $\hat{u} = -2k'\cdot p_q = -2k\cdot p'_q$. Hence $L^{\mu\nu}H_{\mu\nu} = 8(\hat{s}^2+\hat{u}^2)$, and

$$\overline{\sum}|\mathcal{M}|^2 = \frac{e^4 e_q^2}{\hat{t}^{\,2}} \cdot \frac{1}{4}\cdot 8\,(\hat{s}^2+\hat{u}^2) = 2\,e_q^2 e^4\,\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^{\,2}},$$

which is Eq. (1.2.7). ESW obtain the same expression by crossing from $e^+e^- \to q\bar{q}$ [1].

From the amplitude to the cross-section. For two massless particles colliding, the flux factor is $2\hat{s}$. The two-body phase space is $d\Phi_2 = d\cos\theta\, d\phi/32\pi^2$ in the centre-of-mass frame. The matrix element does not depend on the azimuth ϕ, and $\hat{t} = -\tfrac{1}{2}\hat{s}\,(1-\cos\theta)$ gives $d\cos\theta = 2\,d\hat{t}/\hat{s}$, so $d\Phi_2 = d\hat{t}/8\pi\hat{s}$. Therefore

$$\frac{d\hat\sigma}{d\hat t} = \frac{1}{2\hat s}\cdot\frac{1}{8\pi\hat s}\,\overline{\sum}|\mathcal{M}|^2 = \frac{1}{16\pi\hat s^2}\,\overline{\sum}|\mathcal{M}|^2.$$

The result. Put in Eq. (1.2.7), $e^4 = 16\pi^2\alpha^2$, $\hat t = -Q^2$ and $\hat u = -\hat s(1-y)$. Since $d\hat t$ and dQ^2 differ only by a sign that is absorbed in the direction of integration,

$$\frac{d\hat\sigma}{dQ^2} = \frac{1}{16\pi\hat s^2}\cdot 2e_q^2\cdot 16\pi^2\alpha^2\cdot\frac{\hat s^2\,[1+(1-y)^2]}{Q^4} = \frac{2\pi\alpha^2 e_q^2}{Q^4}\,\big[1+(1-y)^2\big],$$

which is Eq. (1.2.9). The check script evaluates the two traces with explicit 4×4 Dirac matrices at two scattering angles and reproduces $8(\hat s^2+\hat u^2)$ to one part in 10^9.

D2. The two time scales in a fast-moving frame

Derives (1.2.5), (1.2.6)

Energy of the proton. With momentum $P \gg M$ along z, $E_P = \sqrt{P^2+M^2} = P + M^2/2P + O(1/P^3)$.

Energy of a configuration of partons. Parton i has longitudinal momentum $\xi_i P$, transverse momentum k_{Ti} and mass m_i. Its energy is

$$E_i = \sqrt{\xi_i^2P^2 + k_{Ti}^2 + m_i^2} = \xi_i P + \frac{k_{Ti}^2+m_i^2}{2\,\xi_i P} + O\Big(\frac{1}{P^3}\Big).$$

The expansion needs $\xi_i P \gg \sqrt{k_{Ti}^2+m_i^2}$. This is why the frame must be fast: every parton with a finite ξ_i then moves forward with a large momentum. Momentum conservation gives $\sum_i \xi_i = 1$ and $\sum_i \vec k_{Ti} = 0$.

The defect. Subtract:

$$\Delta E = \sum_i E_i - E_P = \frac{1}{2P}\left[\sum_i \frac{k_{Ti}^2+m_i^2}{\xi_i} - M^2\right] = \frac{\Delta}{2P}.$$

The terms of order P cancel because $\sum \xi_i = 1$. This is Eq. (1.2.5). A state whose energy differs by ΔE from that of the proton can exist for a time of order $1/\Delta E = 2P/\Delta$.

The photon. In the frame of Eq. (1.4.2), $p = (P,0,0,P)$ and $n = (1/2P,0,0,-1/2P)$. The photon $q = \nu\,n + q_T$ has energy $q^0 = \nu/2P$. With $\nu = Q^2/2x$ this is $Q^2/4xP$. The photon's field oscillates at this frequency, so the absorption is spread over a time of order $1/q^0 = 4xP/Q^2$.

The ratio. $\tau_{\rm int}/\tau_{\rm fluct} = (4xP/Q^2)(\Delta/2P) = 2x\Delta/Q^2$, which is Eq. (1.2.6). Both times grow in proportion to P and the ratio is independent of the frame. The fast frame is the one in which each time can be read as a statement about free particles. The estimate fails where Δ is not small compared with Q^2/x: at low Q^2, and for configurations in which some parton has $\xi_i \to 0$, where the expansion of E_i breaks down.

The numerical example. For the configuration of section 4, $\Delta = 0.18/0.5 + 0.18/0.3 + 0.09/0.2 - 0.938^2 = 0.53$ GeV². With $P = 1000$ GeV the exact energy difference agrees with $\Delta/2P$ to one part in 10^6.

D3. Matching to the structure functions: spin ½ and spin 0

Derives (1.2.12), (1.2.13), (1.2.15), (1.2.16)

Spin ½. Weight Eq. (1.2.11) with the density $q(\xi)$ and sum over flavours. The delta function sets $\xi = x$:

$$\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[1+(1-y)^2\right]\,\frac{1}{2}\sum_{q,\bar q} e_q^2\, q(x).$$

Compare with Eq. (1.2.1). Write $1+(1-y)^2 = y^2 + 2(1-y)$. The functions y^2 and $1-y$ are linearly independent, so their coefficients must agree separately:

$$y^2: \quad F_1 = \frac{1}{2}\sum_{q,\bar q} e_q^2\, q(x), \qquad\qquad (1-y): \quad \frac{F_2}{x} = \sum_{q,\bar q} e_q^2\, q(x).$$

These are Eqs. (1.2.12) and (1.2.13), and their ratio is the Callan–Gross relation (1.2.15).

Spin 0: the amplitude. A charged scalar of charge e_q couples to the photon through the vertex $-ie\,e_q\,V^\mu$ with $V = p_q + p_q'$. The amplitude is $\mathcal{M} = (e^2 e_q/\hat t)\,[\bar u(k')\gamma^\mu u(k)]\,V_\mu$. Only the electron has spin, so the average is a factor $\frac{1}{2}$:

$$\overline{\sum}|\mathcal{M}|^2 = \frac{e^4 e_q^2}{\hat t^{\,2}}\cdot\frac{1}{2}\,L^{\mu\nu}V_\mu V_\nu, \qquad L^{\mu\nu}V_\mu V_\nu = 4\left[2\,(k\cdot V)(k'\cdot V) - V^2\,(k\cdot k')\right].$$

Spin 0: the invariants. $k\cdot V = k\cdot p_q + k\cdot p_q' = \frac{1}{2}(\hat s - \hat u)$, and $k'\cdot V$ has the same value. $V^2 = 2\,p_q\cdot p_q' = -\hat t$ and $k\cdot k' = -\frac{1}{2}\hat t$. With $\hat t = -(\hat s + \hat u)$:

$$L^{\mu\nu}V_\mu V_\nu = 4\Big[\tfrac{1}{2}(\hat s-\hat u)^2 - \tfrac{1}{2}(\hat s+\hat u)^2\Big] = -8\,\hat s\,\hat u = 8\,\hat s^2\,(1-y).$$

Spin 0: the cross-section. With $d\hat\sigma/d\hat t = \overline{\sum}|\mathcal{M}|^2/16\pi\hat s^2$ from D1,

$$\frac{d\hat\sigma}{dQ^2} = \frac{1}{16\pi\hat s^2}\cdot\frac{16\pi^2\alpha^2 e_q^2}{Q^4}\cdot\frac{1}{2}\cdot 8\,\hat s^2(1-y) = \frac{4\pi\alpha^2 e_q^2}{Q^4}\,(1-y).$$

Spin 0: the matching. The mass-shell condition (1.2.10) does not depend on spin, so the delta function $\delta(x-\xi)$ and the incoherent sum are as before. The cross-section now contains $(1-y)$ and no y^2. Comparing with Eq. (1.2.1), the coefficient of y^2 gives $F_1 = 0$ and the coefficient of $(1-y)$ gives $F_2/x = \sum e_q^2\phi(x)$. This is Eq. (1.2.16).

D4. Helicities, and the charged-current cross-sections

Derives (1.2.17), (1.2.27), (1.2.28), (1.2.29)

The centre-of-mass angle. In the electron–quark centre-of-mass frame each particle has energy E, with $\hat s = 4E^2$. If the electron scatters through an angle θ, then $\hat t = -2E^2(1-\cos\theta)$ and $\hat u = -2E^2(1+\cos\theta)$. So $1-y = -\hat u/\hat s = \frac{1}{2}(1+\cos\theta) = \cos^2(\theta/2)$, the first part of Eq. (1.2.17).

Helicity projections. For massless fermions the helicity states are the chirality states, selected by the projectors $P_L = \frac{1}{2}(1-\gamma_5)$ and $P_R = \frac{1}{2}(1+\gamma_5)$. Inserting a projector into a trace of D1 adds an antisymmetric term:

$$\mathrm{Tr}\left[\not k'\gamma^\mu \not k\gamma^\nu P_L\right] = 2\left(k'^\mu k^\nu + k'^\nu k^\mu - g^{\mu\nu}\,k\cdot k'\right) + (\text{a term antisymmetric in } \mu\nu),$$

and the same for the quark, with the opposite sign of the antisymmetric term for P_R. Contracting two such tensors, the symmetric parts give $8\,[(k\cdot p_q)(k'\cdot p_q') + (k\cdot p_q')(k'\cdot p_q)]$ as in D1, and the antisymmetric parts give $\pm 8\,[(k\cdot p_q)(k'\cdot p_q') - (k\cdot p_q')(k'\cdot p_q)]$, with the plus sign when the two helicities are equal. The sums are

$$\text{equal helicities: } 16\,(k\cdot p_q)(k'\cdot p_q') = 4\,\hat{s}^2, \qquad \text{opposite helicities: } 16\,(k\cdot p_q')(k'\cdot p_q) = 4\,\hat{u}^2.$$

There are two combinations of each kind, and $2\cdot 4\hat{s}^2 + 2\cdot 4\hat{u}^2 = 8(\hat{s}^2+\hat{u}^2)$ recovers D1. This is the second part of Eq. (1.2.17). The check script confirms both values with explicit matrices.

The charged current. For $Q^2 \ll M_W^2$ the amplitude for $\nu(k) + d(p_q) \to \mu^-(k') + u(p_q')$ is

$$\mathcal{M} = \frac{G_F}{\sqrt{2}}\,\big[\bar{u}(k')\gamma^\mu(1-\gamma_5)u(k)\big]\,\big[\bar{u}(p_q')\gamma_\mu(1-\gamma_5)u(p_q)\big].$$

Each factor $(1-\gamma_5) = 2P_L$ selects left-handed particles. Summing over spins, each trace is 4 times the projected trace above, so the product is 16 times the equal-helicity result:

$$\sum|\mathcal{M}|^2 = \frac{G_F^2}{2}\cdot 16\cdot 16\,(k\cdot p_q)(k'\cdot p_q') = 32\,G_F^2\,\hat{s}^2.$$

Averaging. The neutrino has one helicity state and the quark two, so the average is a factor $\tfrac{1}{2}$: $\overline{\sum}|\mathcal{M}|^2 = 16\,G_F^2\hat{s}^2$. With $d\hat{\sigma}/d\hat{t} = \overline{\sum}|\mathcal{M}|^2/16\pi\hat{s}^2$ and $|d\hat{t}| = \hat{s}\,dy$:

$$\frac{d\hat{\sigma}}{dy}(\nu q) = \hat{s}\cdot\frac{16\,G_F^2\hat{s}^2}{16\pi\hat{s}^2} = \frac{G_F^2\,\hat{s}}{\pi}.$$

Antiquarks. For $\nu(k) + \bar{u}(p_q) \to \mu^-(k') + \bar{d}(p_q')$ the quark current is $\bar{v}(p_q)\gamma_\mu(1-\gamma_5)v(p_q')$. In the trace the roles of p_q and p_q' are exchanged, which replaces $(k\cdot p_q)(k'\cdot p_q')$ by $(k\cdot p_q')(k'\cdot p_q)$, that is, $\hat{s}^2$ by $\hat{u}^2 = \hat{s}^2(1-y)^2$. This gives the second part of Eq. (1.2.27). Antineutrino cross-sections follow by exchanging quark and antiquark throughout.

The proton. A neutrino can be absorbed by d, s, $\bar{u}$ and $\bar{c}$. Weight each elementary cross-section with the density, set $\xi = x$ by the mass-shell condition, and use $\hat{s} = xs$. The result is Eq. (1.2.28). To read off the structure functions, put $F_2 = 2xF_1$ in the general form quoted in section 9:

$$(1-y)F_2 + \tfrac{1}{2}y^2F_2 \pm y\big(1-\tfrac{1}{2}y\big)xF_3 = \tfrac{1}{2}\big(F_2 \pm xF_3\big) + \tfrac{1}{2}(1-y)^2\big(F_2 \mp xF_3\big).$$

Comparing with Eq. (1.2.28): $\frac{1}{4}(F_2^\nu + xF_3^\nu) = x(d+s)$ and $\frac{1}{4}(F_2^\nu - xF_3^\nu) = x(\bar{u}+\bar{c})$, which is the first line of Eq. (1.2.29). The second line follows in the same way.

D5. The number sum rules from the proton's quantum numbers

Derives (1.2.20)

Let $N_q = \int_0^1 dx\,[q(x) - \bar{q}(x)]$ be the net number of quarks of flavour q. A quark adds $+1$ to it and an antiquark -1, so N_q is the eigenvalue of a conserved flavour charge and is the same for every configuration of partons that the proton can be found in.

Three conserved quantities of the proton are known. Each quark carries baryon number $\frac{1}{3}$, each strange quark strangeness -1, and the charges are $\frac{2}{3}$ and $-\frac{1}{3}$. For three light flavours:

$$\begin{aligned} \text{baryon number:} &\quad \tfrac{1}{3}\,(N_u + N_d + N_s) = 1, \\ \text{charge:} &\quad \tfrac{2}{3}N_u - \tfrac{1}{3}N_d - \tfrac{1}{3}N_s = 1, \\ \text{strangeness:} &\quad -N_s = 0. \end{aligned}$$

The third gives $N_s = 0$. Then $N_u + N_d = 3$ and $2N_u - N_d = 3$. Adding them, $3N_u = 6$, so $N_u = 2$ and $N_d = 1$. These are Eq. (1.2.20). The same argument applied to charm gives $\int (c - \bar{c})\, dx = 0$.

D6. The flavour relations: the ratio bounds and the 5/18 rule

Derives (1.2.24), (1.2.26), (1.2.30), (1.2.32)

The ratio. From Eqs. (1.2.18) and (1.2.23) without charm, $9F_2^p/x = 4U + D + S$ and $9F_2^n/x = U + 4D + S$, which is the first part of Eq. (1.2.24).

The lower bound. Subtract $\frac{1}{4}$ from the ratio and put the result over a common denominator:

$$\frac{F_2^n}{F_2^p} - \frac{1}{4} = \frac{4\,(U + 4D + S) - (4U + D + S)}{4\,(4U + D + S)} = \frac{15D + 3S}{4\,(4U + D + S)} \;\geq\; 0.$$

Equality needs $D = S = 0$.

The upper bound.

$$4 - \frac{F_2^n}{F_2^p} = \frac{4\,(4U + D + S) - (U + 4D + S)}{4U + D + S} = \frac{15U + 3S}{4U + D + S} \;\geq\; 0,$$

with equality only for $U = S = 0$. Both steps use only that U, D and S are non-negative. Including charm changes neither bound, since $c + \bar{c}$ enters numerator and denominator with the same coefficient 4.

The Gottfried sum. Subtracting the two structure functions, $9(F_2^p - F_2^n)/x = 3(U - D)$, the first form of Eq. (1.2.25). With $U = u_v + 2\bar{u}$ and $D = d_v + 2\bar{d}$, $U - D = (u_v - d_v) + 2(\bar{u} - \bar{d})$. Integrating and using $\int (u_v - d_v)\, dx = 2 - 1 = 1$ gives Eq. (1.2.26).

The electromagnetic structure function of an isoscalar target. Average Eqs. (1.2.18) and (1.2.23), now keeping charm and writing $C = c + \bar{c}$:

$$F_2^{eN} = \frac{x}{2}\Big[\frac{5}{9}\,(U + D) + \frac{2}{9}\,S + \frac{8}{9}\,C\Big] = x\Big[\frac{5}{18}\,(U + D) + \frac{1}{9}\,S + \frac{4}{9}\,C\Big].$$

The neutrino structure function. From Eq. (1.2.29), $F_2^{\nu p} = 2x[d + s + \bar{u} + \bar{c}]$. Isospin exchanges u and d, so $F_2^{\nu n} = 2x[u + s + \bar{d} + \bar{c}]$. The average is $x[u + \bar{u} + d + \bar{d} + 2s + 2\bar{c}]$. With $s = \bar{s}$ and $c = \bar{c}$ this is $x[U + D + S + C] = x\Sigma$, the first part of Eq. (1.2.30).

The ratio. Write $U + D = \Sigma - S - C$:

$$F_2^{eN} = x\Big[\frac{5}{18}\Sigma - \Big(\frac{5}{18} - \frac{2}{18}\Big)S + \Big(\frac{8}{18} - \frac{5}{18}\Big)C\Big] = \frac{5}{18}\,x\,\Sigma\Big[1 - \frac{3}{5}\,\frac{S - C}{\Sigma}\Big].$$

Dividing by $F_2^{\nu N} = x\Sigma$ gives Eq. (1.2.32).

The valence combination. From Eq. (1.2.29) and isospin, $xF_3^{\nu N} = x[u_v + d_v + 2s - 2\bar{c}]$ and $xF_3^{\bar{\nu} N} = x[u_v + d_v - 2\bar{s} + 2c]$. Their average is $x[u_v + d_v + (s - \bar{s}) + (c - \bar{c})]$, which is the second part of Eq. (1.2.30) for $s = \bar{s}$, $c = \bar{c}$. Integrating with Eq. (1.2.20) gives Eq. (1.2.31); the strange and charm terms integrate to zero even if $s \neq \bar{s}$ point by point.

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