1.1 Deep inelastic scattering

Kinematics, the cross-section, and the structure functions that hold everything the photon can learn about the proton.

1. The experiment

Deep inelastic scattering is the measurement on which the whole subject rests. A beam of leptons hits a proton, and only the scattered lepton is recorded:

$$\ell(k) + p(P) \longrightarrow \ell'(k') + X.$$

The letters in brackets are four-momenta. X stands for whatever hadrons come out. It is not measured, and the cross-section is summed over every possibility. A measurement of this kind is called *inclusive*.

A lepton is used as the projectile for two reasons. It has no structure of its own down to the smallest distances yet probed, and its interaction is electroweak, which is weak enough to be treated at lowest order and is known exactly. Everything in the cross-section that is not understood therefore belongs to the proton.

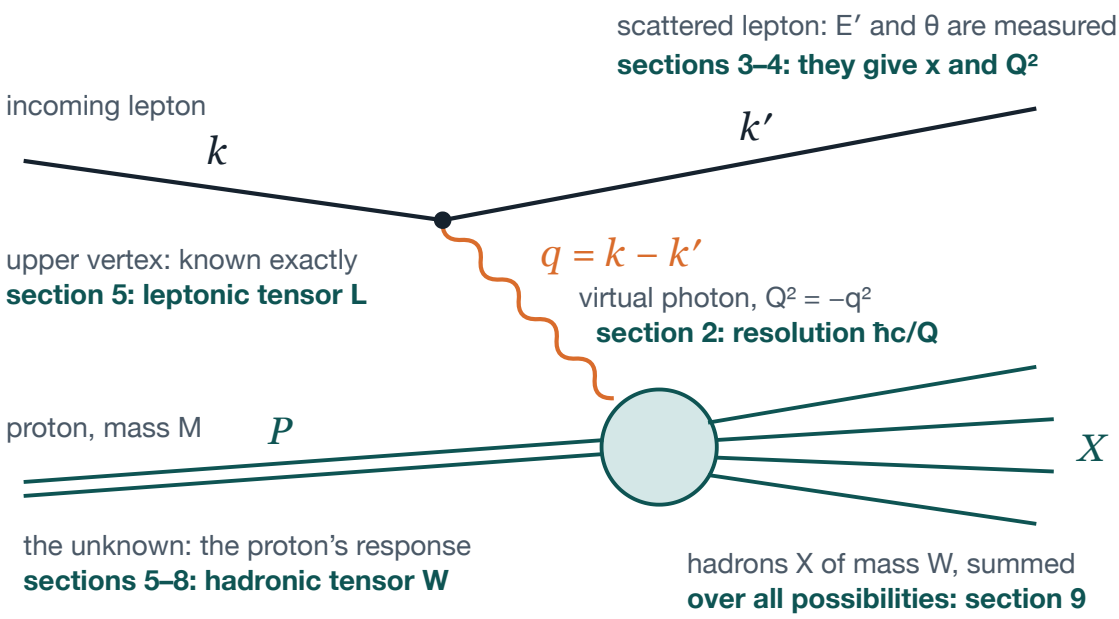


Figure 1. The chapter in one picture: inclusive lepton–proton scattering at lowest order in the electromagnetic coupling. The shaded circle is the unknown: the response of the proton to a photon of momentum q. The bold labels name the section that treats each part.

The argument of the chapter, in six lines:

1. The photon's virtuality Q^2 sets the size of what it can resolve (section 2).
2. Each event is two numbers, x and Q^2, read off the energy and angle of the scattered lepton (sections 3 and 4).
3. The cross-section is a known lepton factor times an unknown proton factor, the hadronic tensor $W^{\alpha\beta}$ (section 5).

4. Symmetry leaves the proton factor two functions, $F_1(x, Q^2)$ and $F_2(x, Q^2)$, each entering the cross-section with a known weight (sections 6 and 7).
5. They are the proton's responses to transversely and longitudinally polarised photons (section 8).
6. Struck softly, the proton recoils intact or is excited to a resonance. Struck hard, it breaks up, and that cross-section does not fade as Q^2 grows (sections 9 and 11). W or Z exchange adds a third function (section 10).

No model of the proton is used anywhere in the chapter. The structure functions are defined by the measurement. What they mean in terms of quarks is the subject of chapters 1.2 and 1.3.

The treatment follows Ellis, Stirling and Webber, section 4.1 and the opening of section 4.2 [1], and Moore's thesis, section 1.1.1 [2]. Every formula has been checked numerically, including the traces with explicit Dirac matrices; the script is linked at the end.

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2. The photon as a probe: what the virtuality resolves

A wave cannot show detail finer than its wavelength. The exchanged photon is a wave in this sense, and its wavelength is set by the momentum it carries: the harder the lepton is deflected, the finer the detail. Figure 2 draws the wave to scale against the proton at three momentum transfers.

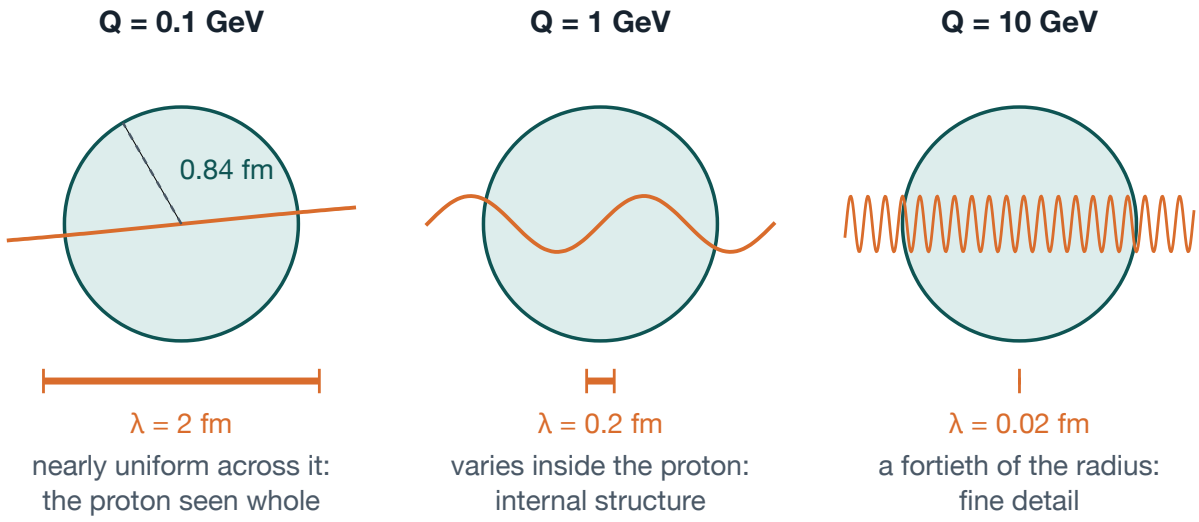


Figure 2. The exchanged photon as a wave, drawn to scale against a proton of charge radius 0.84 fm. Each orange bar is the reduced wavelength λ of Eq. (1.1.2), and the wave has period $2\pi\lambda$. At 10 GeV the bar is too short to see at this scale.

The measure of the deflection is the square of the photon's four-momentum $q = k - k'$. Let E and E' be the lepton's energy before and after the scattering and θ the angle through which it is deflected, all in one frame. For a lepton of negligible mass (derivation D1)

$$Q^2 \equiv -q^2 = 2\,k \cdot k' = 4EE' \sin^2\frac{\theta}{2} \geq 0. \tag{1.1.1}$$

Q^2 is called the photon's virtuality. It is large when the lepton is deflected through a large angle and still leaves with a large energy. It is never negative, so the photon's momentum is *space-like*.

Because q is space-like, there is a frame in which it has no energy component at all. This is the Breit frame, in which $q^\mu = (0, 0, 0, -Q)$. There the photon is a static field that varies in space along the collision axis with a reduced wavelength

$$\lambda = \frac{\hbar c}{Q} = \frac{0.197\ \text{fm}}{Q/\text{GeV}}. \tag{1.1.2}$$

One number is worth remembering: 1 GeV corresponds to 0.2 fm, a quarter of the proton's charge radius of about 0.84 fm.

MOMENTUM SCALE	WAVELENGTH	WHAT THE PHOTON SEES
$Q = 0.1$ GeV	$\lambda = 2$ fm	the proton as a whole: its charge and magnetic moment
1 GeV	0.2 fm	a quarter of the proton's radius: internal structure
10 GeV	0.02 fm	a fortieth of the radius
175 GeV	0.001 fm	near the top of the range measured at the HERA collider, $Q^2 \approx 3 \times 10^4\ \text{GeV}^2$; the published data extend to $5 \times 10^4\ \text{GeV}^2$

In hand: the meaning of Q^2. It is the resolution of the probe. Raising it turns up the magnification, and elastic and deep inelastic scattering are what the same experiment shows at low and at high magnification.

3. The invariants

The scattered lepton has an energy and an angle, so an inclusive event has two degrees of freedom. They are best expressed as two frame-independent numbers. Q^2 is one. The second can be the fraction of its energy the lepton gave up (y), the mass of the hadronic debris (W), or a ratio x that proves to be the natural variable for the proton's structure. Figure 3 shows them together: an event is a point on a map, and each variable is a family of lines on it.

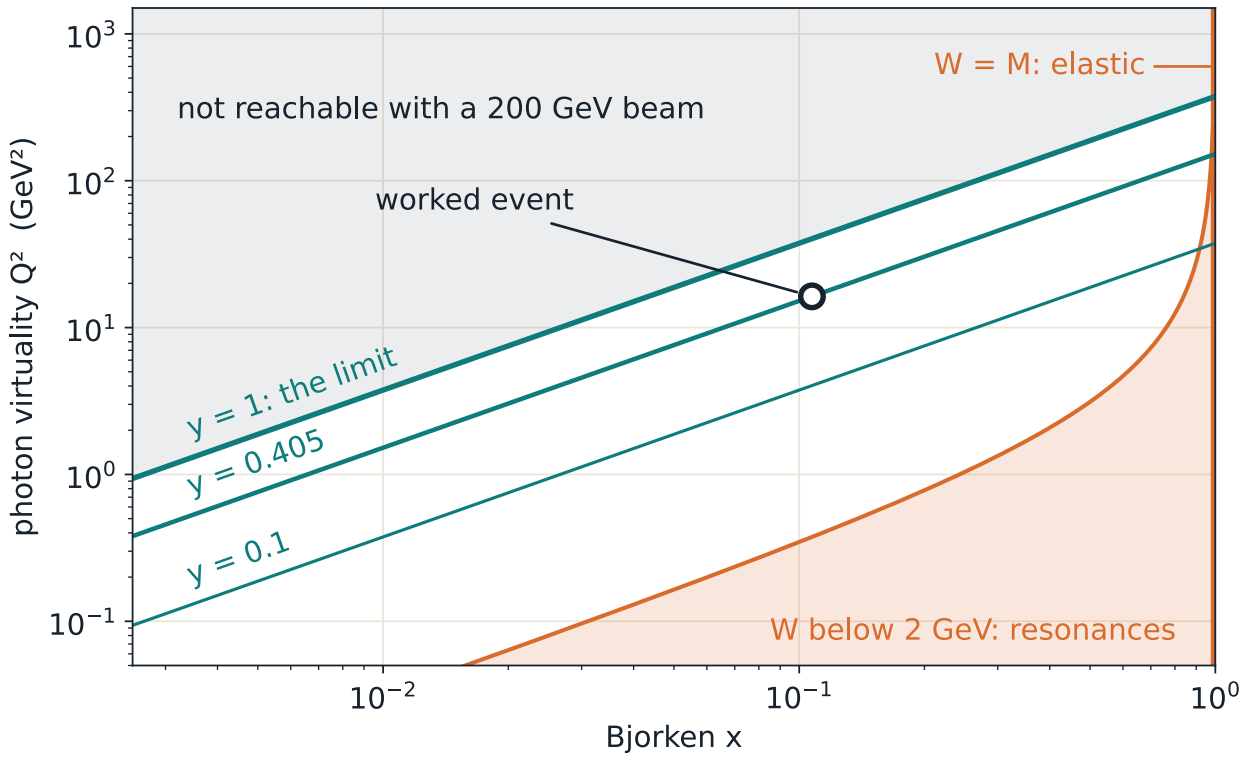


Figure 3. The kinematic plane for 200 GeV muons on protons at rest, $s = 376.2\ \mathrm{GeV}^2$, on logarithmic axes. Lines of constant y are parallel diagonals, Eq. (1.1.5), and nothing lies above $y = 1$. Lines of constant hadronic mass W, Eq. (1.1.4), bend upwards towards the edge $x = 1$, which is elastic scattering. The circle is the fixed-target event worked in section 4. All lines are computed.

Definitions

The momenta k, k' and P form three independent scalar products. One is fixed by the beam, and two describe the event. The standard choices are (algebra in derivation D1)

$$\nu = P \cdot q, \qquad x = \frac{Q^2}{2\nu}, \qquad y = \frac{P \cdot q}{P \cdot k}, \tag{1.1.3}$$

together with Q^2 and the two invariant masses

$$s = (k + P)^2, \qquad W^2 = (P + q)^2 = M^2 + 2\nu - Q^2 = M^2 + Q^2\,\frac{1 - x}{x}. \tag{1.1.4}$$

Each has a direct meaning.

- ν is the energy the lepton loses, up to a factor. In the proton's rest frame $P = (M, \mathbf{0})$, so $\nu = M(E - E')$. Note the convention: in this guide, as in ESW, ν is the invariant $P \cdot q$ and has dimensions of mass squared. Much of the older literature uses ν for the energy loss $E - E'$ itself.
- y is the fraction of the lepton's energy that is transferred, $y = 1 - E'/E$ in the proton's rest frame. It is called the inelasticity.
- s is the squared centre-of-mass energy of the lepton and proton. It is set by the accelerator.
- W^2 is the squared invariant mass of the hadronic final state X, since $P + q$ is its total momentum. By the last form in Eq. (1.1.4), at fixed Q^2 the debris is heavier the smaller x is, and at $x = 1$ it is a single proton.

- x is the Bjorken variable. It is dimensionless and, so far, only a ratio of two invariants. Chapter 1.2 shows that it is the fraction of the proton's momentum carried by the struck quark. Nothing in this chapter depends on that.

One relation and two ranges

Five variables describe an event with two degrees of freedom, so three relations connect them. Two are the definition of x and Eq. (1.1.4). The third follows from $s - M^2 = 2P \cdot k$:

$$Q^2 = x\,y\,(s - M^2) \;\approx\; x\,y\,s. \tag{1.1.5}$$

On logarithmic axes this is a straight line of unit slope for each y: the diagonals of Figure 3. At a fixed beam energy y is therefore not independent, and an event is a point in the (x, Q^2) plane.

The ranges follow from two physical requirements. The final state X has baryon number one, so it is at least as heavy as a proton: $W^2 \geq M^2$. And the scattered lepton cannot be deflected by more than 180°. Together (derivation D1):

$$0 < x \leq 1, \qquad 0 < y \leq \frac{1}{1 + M^2 x/(s - M^2)}. \tag{1.1.6}$$

The point $x = 1$ is elastic scattering, $W = M$. Everything inelastic lies at $x < 1$. The upper limit on y is 1 when the proton mass is neglected. Since $y \leq 1$, Eq. (1.1.5) gives $Q^2 \leq x\,(s - M^2)$: the largest virtuality an accelerator can reach at a given x is set by its s. This is the edge of the grey region in Figure 3.

The same kinematics in light-cone vectors

Chapter 1.4 writes the momenta in terms of two light-like vectors p and n with $n \cdot p = 1$, Eq. (1.4.2). With the proton mass kept, the decomposition is

$$P^\mu = p^\mu + \frac{M^2}{2}\,n^\mu, \qquad q^\mu = \nu\,n^\mu + q_T^\mu, \qquad q_T^2 = Q^2, \tag{1.1.7}$$

where q_T is a transverse vector and q_T^2 denotes its positive squared length. These reproduce $P^2 = M^2$, $q^2 = -Q^2$ and $P \cdot q = \nu$ (derivation D1). The ν and x of this chapter are therefore exactly those of chapter 1.4, and neglecting M is the replacement $P \to p$.

In hand: two coordinates, x and Q^2, for every event, with y and W as lines on the same map.

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4. From the detector to the kinematic plane

An experiment records energies and angles in its own frame. This section turns the scattered lepton's energy E' and deflection angle θ into x and Q^2 for the two kinds of experiment, and works one event of each. Figure 4 shows the two events as they look in the laboratory.

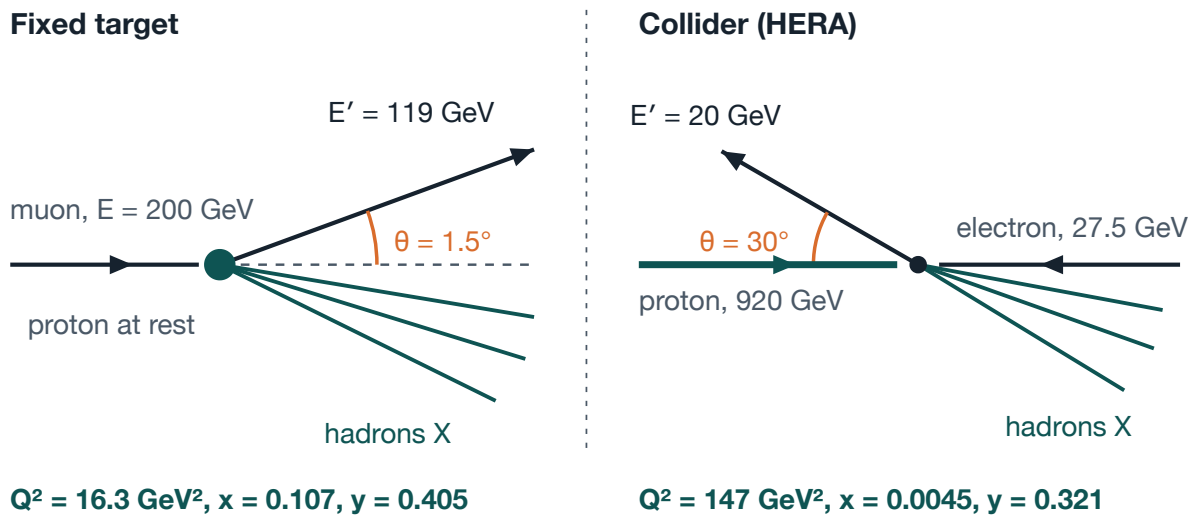


Figure 4. The two worked events as a detector sees them. Left: a muon beam on a proton at rest; the 1.5° deflection is drawn larger than it is. Right: head-on beams at HERA; the 30° deflection is to scale. The scattered lepton alone fixes the event.

Fixed target

A beam of energy E strikes a proton at rest. With $P = (M, \mathbf{0})$ the invariants of section 3 become

$$Q^2 = 4EE'\sin^2\frac{\theta}{2}, \qquad y = 1 - \frac{E'}{E}, \qquad x = \frac{Q^2}{2M(E - E')}, \qquad s = M^2 + 2ME. \tag{1.1.8}$$

A small angle means a small Q^2. A large energy loss means a large y and, at fixed Q^2, a small x.

Collider

A lepton of energy E_e meets a proton of energy $E_p \gg M$ head-on. Let θ again be the angle between the lepton's incoming and outgoing directions. Then (derivation D1)

$$Q^2 = 4E_eE'\sin^2\frac{\theta}{2}, \qquad y = 1 - \frac{E'}{E_e}\cos^2\frac{\theta}{2}, \qquad x = \frac{Q^2}{s\,y}, \qquad s = 4E_eE_p. \tag{1.1.9}$$

The experiments at HERA quote the lepton's polar angle from the proton beam direction, which is $\pi - \theta$. Collider detectors also see the hadrons, and x and Q^2 can be reconstructed from them as well. That is how charged-current events, whose outgoing lepton is an unseen neutrino, are placed on the plane.

The important difference is in s. For a fixed target s grows linearly with the beam energy. For a collider it grows as the product of the two. HERA collided 27.5 GeV electrons or positrons with 920 GeV protons, which gives $s = 1.012 \times 10^5$ GeV^2. A fixed-target experiment would need a beam of 54,000 GeV to match it.

Two worked events

	FIXED TARGET	COLLIDER
Beam	$E = 200$ GeV muons on protons at rest	$E_e = 27.5$ GeV on $E_p = 920$ GeV
Measured	$E' = 119$ GeV, $\theta = 1.5^\circ$	$E' = 20$ GeV, $\theta = 30^\circ$
s	$376.2\ \text{GeV}^2$	$1.012 \times 10^5\ \text{GeV}^2$
Q^2	$4 \cdot 200 \cdot 119 \cdot \sin^2(0.75^\circ) = 16.3\ \text{GeV}^2$	$4 \cdot 27.5 \cdot 20 \cdot \sin^2(15^\circ) = 147\ \text{GeV}^2$
y	$1 - 119/200 = 0.405$	$1 - (20/27.5)\cos^2(15^\circ) = 0.321$
x	$16.3/(2 \cdot 0.938 \cdot 81) = 0.107$	$147.4/(101200 \cdot 0.3214) = 0.0045$
W	11.7 GeV	180 GeV
Resolution λ	0.049 fm	0.016 fm

Both events are far above the resonance region. In the first, Eq. (1.1.5) gives a check: $x\,y\,(s - M^2) = 0.1073 \times 0.405 \times 375.3 = 16.3\ \text{GeV}^2$. The collider event has a similar y but a value of x more than twenty times smaller.

The kinematic plane

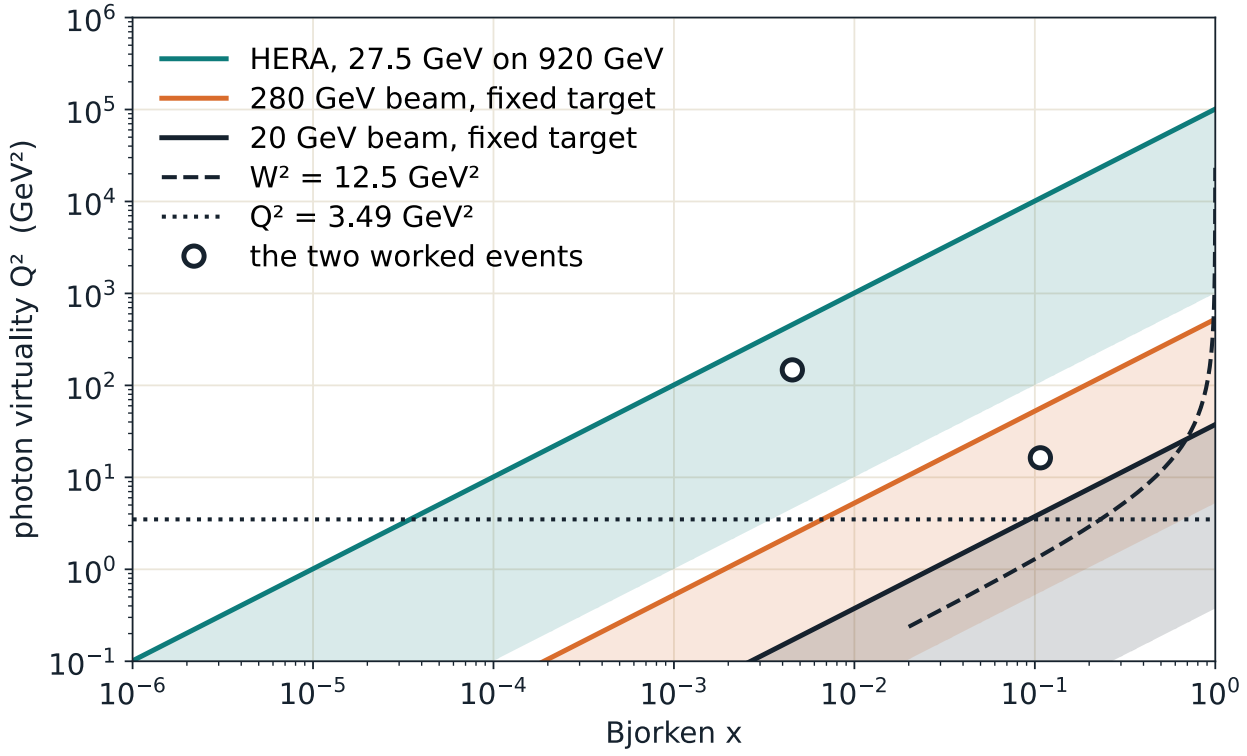


Figure 5. Where the measurements can lie. Each solid line is the limit $y = 1$ of Eq. (1.1.5), $Q^2 = x\,(s - M^2)$, for one accelerator, and the shaded band below it covers $0.01 < y < 1$. The HERA beams are 27.5 GeV electrons or positrons on 920 GeV protons. The 20 GeV beam is that of the first SLAC experiments, and 280 GeV is the highest muon energy used at CERN by BCDMS and NMC. The circles are the two worked events; the fixed-target one used a 200 GeV beam. The dashed curve is $W^2 = 12.5\ \mathrm{GeV}^2$ from Eq. (1.1.4), and the dotted line is $Q^2 = 3.49\ \mathrm{GeV}^2$. Our fits keep only data above the dotted line and to the left of the dashed curve (section 9). The lines are computed, not drawn; real experiments fill only part of each band.

Figure 5 shows the reach of the two kinds of experiment. Fixed-target data lie at $x \gtrsim 0.01$ and moderate Q^2. HERA extends the reach in Q^2 at fixed x by a factor of about 200 over the 280 GeV beam, and at $Q^2 = 3.49\ \mathrm{GeV}^2$ it reaches down to $x = 3.4 \times 10^{-5}$. The two sets overlap in x but at different Q^2, and comparing them is the test of the evolution equations of chapter 1.4.

In hand: the recipe that places a measured event on the plane, and the region each accelerator can reach.

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5. The cross-section as a product of two tensors

One photon is exchanged, so the process has two halves joined by a single line. A probability is an amplitude times its complex conjugate, and Figure 6 draws that product. The upper half involves only the lepton and can be calculated. The lower half involves only the proton and whatever it turns into. The cross-section is the product of the two.

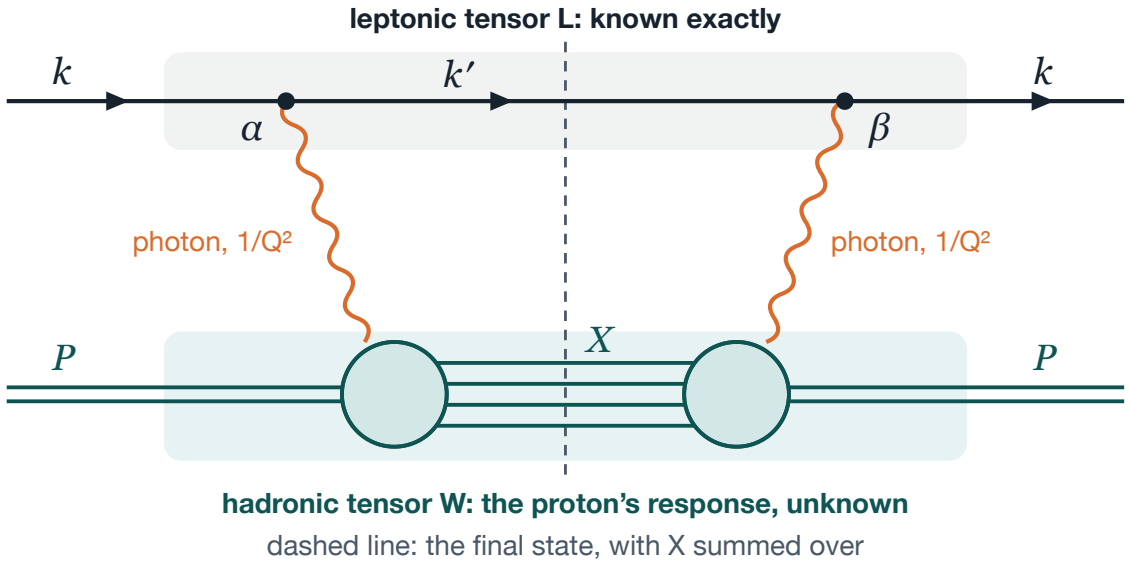


Figure 6. The squared amplitude. Left of the dashed line is the amplitude of Figure 1, right of it its complex conjugate. The upper band is the leptonic tensor $L^{\alpha\beta}$ of Eq. (1.1.11); α and β label the two photon vertices. The lower band is the hadronic tensor $W_{\alpha\beta}$ of Eq. (1.1.12): the proton absorbs the photon, becomes any state X, and is restored. The two photon lines give $1/Q^4$.

The amplitude

The lepton couples to the photon through its current $\bar{u}(k')\gamma^\alpha u(k)$, where u is a Dirac spinor. The proton couples through the electromagnetic current of the quarks, $j_\alpha = \sum_q e_q\, \bar{q}\gamma_\alpha q$, with e_q the quark charge in units of the coupling e. The amplitude for producing a particular final state X is

$$\mathcal{M} = e^2\, \bar{u}(k')\gamma^\alpha u(k)\, \frac{1}{q^2}\, \langle X|\, j_\alpha(0)\, |P\rangle. \tag{1.1.10}$$

The three factors are the three pieces of Figure 1: the lepton vertex, the photon line, and the shaded circle. The last is a matrix element of the current between the proton and the state X. It involves the strong interaction to all orders and cannot be calculated in perturbation theory.

The two tensors

The amplitude is a product, so its square is a product. Averaging over the spin of the incoming lepton and summing over the spin of the outgoing one gives the *leptonic tensor* (derivation D2):

$$L^{\alpha\beta} = \frac{1}{2}\,\mathrm{Tr}\big[k\!\!\!/'\gamma^\alpha k\!\!\!/\gamma^\beta\big] = 2\big(k^\alpha k'^\beta + k^\beta k'^\alpha - g^{\alpha\beta}\, k\cdot k'\big). \tag{1.1.11}$$

It contains nothing about the target. It is symmetric and obeys $q_\alpha L^{\alpha\beta} = 0$, which expresses conservation of the lepton's current.

Everything on the proton's side is collected into the *hadronic tensor*. It is the squared matrix element of the current, averaged over the proton's spin and summed over all final states and their phase space:

$$W_{\alpha\beta}(P,q) = \frac{1}{4\pi}\sum_X \langle P|\, j^\dagger_\beta(0)\, |X\rangle\langle X|\, j_\alpha(0)\, |P\rangle\, (2\pi)^4\delta^4(P+q-p_X). \tag{1.1.12}$$

Read it from the right. The current turns the proton into X, the conjugate factor turns it back, and the sum runs over everything the proton can become: a single proton, a resonance, or a spray of hadrons. The $1/4\pi$ is a convention, the one used by ESW and by chapter 1.4.

Because every final state is summed over, the sum can be done with the completeness relation, and the tensor becomes a matrix element in the proton state alone (derivation D3):

$$W_{\alpha\beta}(P,q) = \frac{1}{4\pi}\int d^4z\, e^{iq\cdot z}\,\langle P|\, j_\beta^\dagger(z)\, j_\alpha(0)\,|P\rangle. \tag{1.1.13}$$

Equation (1.1.13) states what inclusive scattering measures: the correlation between the quark current at two space-time points inside the proton, Fourier-transformed in their separation. This form is the starting point for chapter 1.3.

The master formula

Putting the squared amplitude together with the flux factor and the lepton's phase space, and changing variables from the lepton's momentum to x and y (derivation D5):

$$\frac{d^2\sigma}{dx\,dy} = \frac{2\pi\alpha^2\,y}{Q^4}\,L^{\alpha\beta}\,W_{\alpha\beta}, \qquad \alpha = \frac{e^2}{4\pi}. \tag{1.1.14}$$

The factor α^2/Q^4 is two couplings and two photon propagators. It makes every such cross-section fall steeply with Q^2, whatever the target.

> **The separation.** One-photon exchange splits the cross-section into $L^{\alpha\beta}$, which is known, and $W_{\alpha\beta}$, which contains all the strong-interaction physics. The split holds for any target and any final state, and it holds to all orders in the strong coupling. It fails only at higher order in α, where two photons are exchanged or the lepton radiates; experiments correct their data for those effects before quoting structure functions.

In hand: the cross-section as a known tensor contracted with an unknown one, Eq. (1.1.14).

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6. Two structure functions

The hadronic tensor has sixteen components, but the proton cannot fill them independently. Once its spin has been averaged over, the only directions available are its momentum P and the photon's momentum q. A two-index tensor can be built from these in six ways. Symmetry removes or fixes four of the six, and two functions survive. The table shows the count, and derivation D4 does the algebra.

BUILDING BLOCK	FATE	REMOVED OR FIXED BY
$g^{\alpha\beta}$	free: becomes F_1	
$P^\alpha P^\beta$	free: becomes F_2	
$q^\alpha q^\beta$	fixed by the two above	current conservation
$P^\alpha q^\beta + q^\alpha P^\beta$	fixed by the two above	current conservation
$P^\alpha q^\beta - q^\alpha P^\beta$	zero	current conservation
$\epsilon^{\alpha\beta\rho\sigma} P_\rho q_\sigma$	zero	parity

The argument

Each coefficient depends on the two scalars ν and Q^2. Three conditions act on them.

- **Parity.** The electromagnetic current is a vector and the strong interaction conserves parity. The block built with the antisymmetric symbol $\epsilon^{\alpha\beta\rho\sigma}$ changes sign under reflection and is excluded.
- **Current conservation.** The current obeys $\partial^\alpha j_\alpha = 0$, which gives $q^\alpha W_{\alpha\beta} = 0$ and $q^\beta W_{\alpha\beta} = 0$. These remove one block and fix two.
- **Hermiticity.** $W^*_{\alpha\beta} = W_{\beta\alpha}$ makes the two surviving coefficients real.

The result is

$$W^{\alpha\beta}(P,q) = \Big(-g^{\alpha\beta} + \frac{q^\alpha q^\beta}{q^2} \Big) F_1(x,Q^2) + \hat{P}^\alpha \hat{P}^\beta \, \frac{F_2(x,Q^2)}{\nu}, \qquad \hat{P}^\alpha = P^\alpha + \frac{q^\alpha}{2x}. \tag{1.1.15}$$

The two free blocks appear with their fixed companions attached. The vector $\hat{P}$ is the part of P orthogonal to q: $\hat{P} \cdot q = \nu - Q^2/2x = 0$. Each of the two structures vanishes when contracted with q, so current conservation is manifest.

F_1 and F_2 are the *structure functions* of the proton. They are dimensionless, real, and functions of two variables, chosen here as x and Q^2. They are defined by Eq. (1.1.15) and hence by the measurement. In the older notation $F_1 = W_1$ and $F_2 = \nu W_2$.

Projections

Contracting with the light-cone vectors of Eq. (1.1.7) picks out one function at a time (derivation D4):

$$F_2 = \nu \, n_\alpha n_\beta \, W^{\alpha\beta}, \qquad F_L = \frac{4x^2}{\nu} \, p_\alpha p_\beta \, W^{\alpha\beta} \quad (M = 0). \tag{1.1.16}$$

The first is Eq. (1.4.4) of chapter 1.4, and it is exact in the proton mass. The second defines a third function, which is not independent of the other two:

$$F_L = F_2 - 2xF_1 \quad (M = 0), \qquad F_L = \Big(1 + \frac{4M^2x^2}{Q^2}\Big) F_2 - 2xF_1 \quad (\text{in general}). \tag{1.1.17}$$

F_L is the *longitudinal structure function*. Section 8 shows that it measures the absorption of longitudinally polarised photons, which explains the name and fixes the form of the mass term.

In hand: everything a photon can learn about an unpolarised proton is two functions of x and Q^2.

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7. The cross-section in terms of the structure functions

Contracting the two tensors gives the formula used to extract structure functions from data. Its shape is simple: a prefactor that is the same for any target, times the two structure functions, each with a known weight that depends on y. Figure 7 shows the weights. At small y the cross-section measures F_2 almost alone. The proton mass is kept throughout the section.

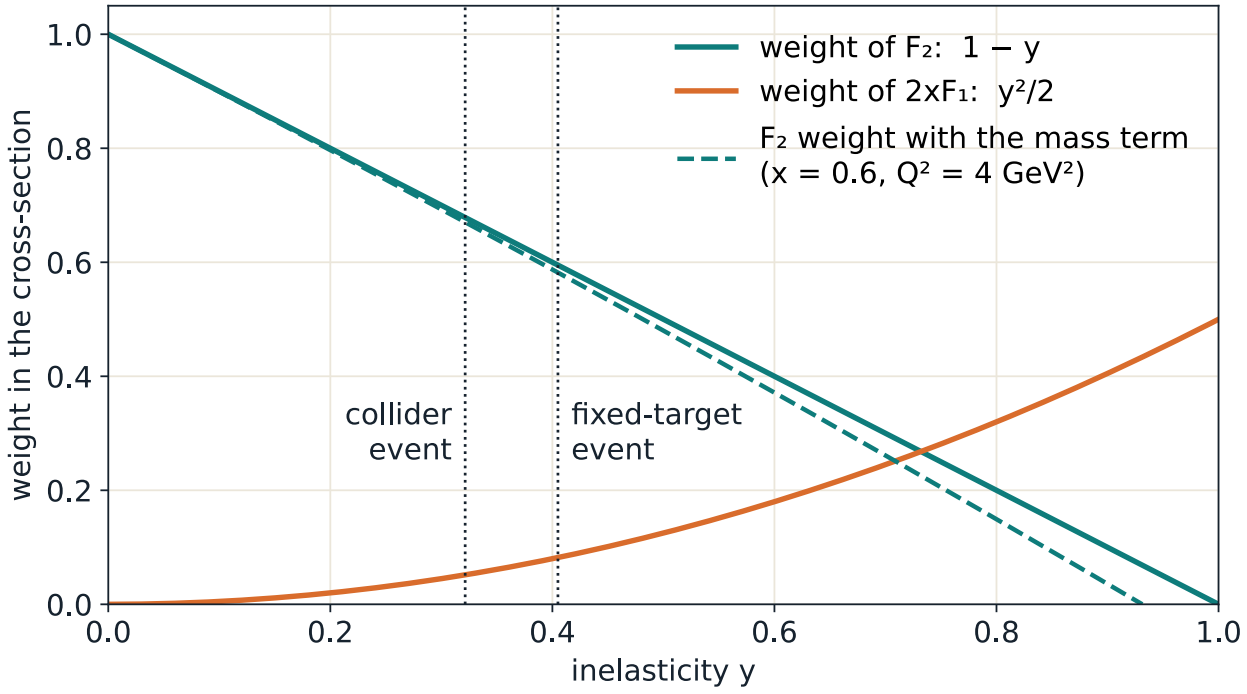


Figure 7. The weights that multiply the structure functions in the bracket of Eq. (1.1.19), for a massless proton: $1-y$ for F_2, and $y^2/2$ for the combination $2xF_1$. They are equal at $y = 0.73$. The dashed curve keeps the proton mass, at $x = 0.6$ and $Q^2 = 4\ \mathrm{GeV}^2$. At the worked events of section 4 the two weights are 0.595 and 0.082 (fixed target) and 0.679 and 0.052 (collider).

The contraction

Because $q_\alpha L^{\alpha\beta} = 0$, every term of Eq. (1.1.15) that contains q^α or q^β drops out. What remains is $-g^{\alpha\beta}F_1 + P^\alpha P^\beta F_2/\nu$, and the contraction needs only $P\cdot k$ and $P\cdot k'$ in terms of the invariants (derivation D5):

$$L^{\alpha\beta}W_{\alpha\beta} = \frac{2Q^2}{x\,y^2}\left[x\,y^2 F_1 + \left(1 - y - \frac{M^2x^2y^2}{Q^2}\right)F_2\right]. \tag{1.1.18}$$

The result

Inserting this into Eq. (1.1.14):

$$\frac{d^2\sigma}{dx\,dy} = \frac{4\pi\alpha^2}{x\,y\,Q^2}\left[x\,y^2\,F_1(x,Q^2) + \left(1 - y - \frac{M^2x^2y^2}{Q^2}\right)F_2(x,Q^2)\right]. \tag{1.1.19}$$

The bracket holds the proton: two unknown functions with the weights of Figure 7. This is Moore's Eq. (1.22), and with $Q^2 = 2MExy$ for a fixed target it is ESW's Eq. (4.2). At fixed x, Eq. (1.1.5) gives $dQ^2 = (Q^2/y)\,dy$, so

$$\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{x\,Q^4}\left[x\,y^2\,F_1 + \left(1 - y - \frac{M^2x^2y^2}{Q^2}\right)F_2\right]. \tag{1.1.20}$$

With the mass neglected and F_1 traded for F_L through Eq. (1.1.17), this takes the form in which the HERA experiments present their results:

$$\frac{d^2\sigma}{dx\,dQ^2} = \frac{2\pi\alpha^2}{x\,Q^4}\Big[Y_+\,F_2 - y^2F_L\Big], \qquad Y_+ = 1 + (1-y)^2 \qquad (M = 0). \tag{1.1.21}$$

Y_+ lies between 1 and 2, and y^2 is small except at large y: F_2 is the main term and F_L a correction.

In terms of what a fixed-target experiment records, the energy E' and the solid angle $d\Omega$ of the scattered lepton, the same formula reads

$$\frac{d^2\sigma}{dE'\,d\Omega} = \frac{\alpha^2\cos^2\frac{\theta}{2}}{4E^2\sin^4\frac{\theta}{2}}\left[\frac{F_2}{E - E'} + \frac{2F_1}{M}\tan^2\frac{\theta}{2}\right]. \tag{1.1.22}$$

The prefactor is the Mott cross-section, which describes scattering from a fixed point charge. The bracket is the proton's modification of it.

What the formula says

- **Two functions, two different dependences on y.** By Eq. (1.1.5), changing y at fixed x and Q^2 means changing s. Measuring the same (x, Q^2) point at two or more beam energies therefore separates F_1 from F_2.
- **F_2 dominates.** Over most of the range y is small and the bracket is close to F_2. Most data are published as F_2, or as a *reduced cross-section* $\sigma_r = F_2 - (y^2/Y_+)F_L$, which is Eq. (1.1.21) with the kinematic prefactor divided out.
- **The mass term.** The coefficient of F_2 equals $(E'/E)\cos^2(\theta/2)$ in the proton's rest frame (derivation D5), so it is never negative, and it vanishes for backward scattering. The term $M^2x^2y^2/Q^2$ matters at large x and low Q^2, where the fixed-target data sit. It is the kinematic part of the *target-mass corrections*. The other part, the dependence of the structure functions themselves on M^2/Q^2, is not derived here.

In hand: the formula that turns a measured cross-section into F_2, and the reason F_1 needs a second beam energy.

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8. Transverse and longitudinal photons

A photon is a field with a direction of oscillation, its polarisation. For a real photon the field oscillates across the direction of travel, in one of two ways. A virtual photon can also oscillate along it. The proton responds differently to the two kinds, and the responses are the structure functions in another guise: F_1 for transverse photons and F_L for longitudinal ones. The lepton sets the mixture, through y. Figure 8 shows both points.

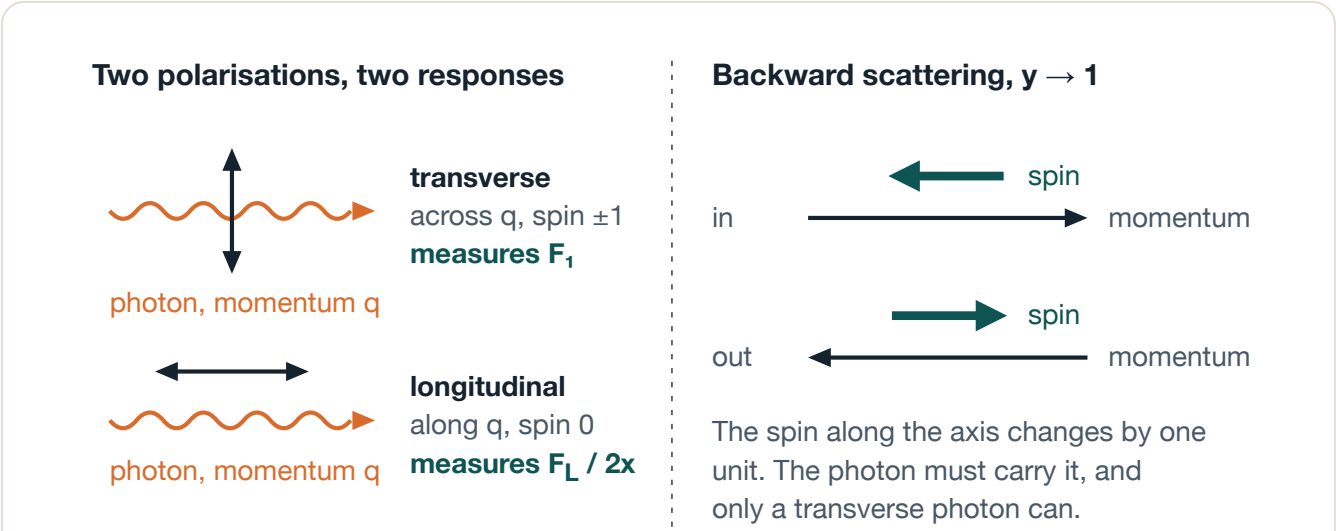


Figure 8. Left: the two kinds of polarisation of a virtual photon of momentum q, and the structure function that measures the proton's response to each, Eq. (1.1.23). Right: scattering straight back in the centre-of-mass frame, $y \to 1$. A massless lepton keeps its helicity, so its spin along the collision axis reverses with its motion.

The photon's polarisation states

Let $\varepsilon_\pm$ be the two transverse polarisation vectors and ε_L the longitudinal one, all orthogonal to q. For a space-like photon the longitudinal vector has $\varepsilon_L^2 = +1$ and is proportional to the vector $\hat{P}$ of Eq. (1.1.15). Rewriting that equation in terms of these vectors (derivation D6) gives

$$W^{\alpha\beta} = F_1 \sum_{\lambda=\pm} \varepsilon_\lambda^\alpha \varepsilon_\lambda^{\beta *} + \frac{F_L}{2x}\, \varepsilon_L^\alpha \varepsilon_L^\beta, \qquad \varepsilon_L^\alpha = \frac{\hat{P}^\alpha}{\sqrt{\hat{P}^2}}, \tag{1.1.23}$$

with F_L the general form in Eq. (1.1.17). This is exact. No term mixes one polarisation with another: F_1 is the response to a transverse photon and $F_L/2x$ the response to a longitudinal one.

Dividing each response by a flux factor turns it into the cross-section for the proton to absorb a photon of that polarisation. For a virtual photon the flux is a matter of convention. With the common choice $\nu(1-x)$ (derivation D6),

$$\sigma_T = \frac{4\pi^2\alpha}{\nu(1-x)}\, F_1, \qquad \sigma_L = \frac{4\pi^2\alpha}{\nu(1-x)}\, \frac{F_L}{2x}, \qquad R \equiv \frac{\sigma_L}{\sigma_T} = \frac{F_L}{2xF_1}. \tag{1.1.24}$$

The ratio R does not depend on the flux convention. Both cross-sections are positive, so $F_1 \geq 0$ and $F_L \geq 0$. For $M = 0$ this gives $0 \leq 2xF_1 \leq F_2$.

The lepton as a source of photons

Contracting the leptonic tensor with the same polarisation vectors gives the rate at which the lepton supplies photons of each kind. The cross-section becomes (derivation D6)

$$\frac{d^2\sigma}{dx\, dQ^2} = \Gamma \left[\sigma_T + \varepsilon\, \sigma_L\right], \qquad \Gamma = \frac{\alpha\,(1-x)\,y^2}{2\pi\, x\, Q^2\,(1-\varepsilon)}. \tag{1.1.25}$$

Γ is the flux of transverse photons radiated by the lepton, and ε is the ratio of the longitudinal flux to the transverse flux:

$$\varepsilon = \frac{1 - y - \frac{1}{4}\gamma^2 y^2}{1 - y + \frac{1}{2}y^2 + \frac{1}{4}\gamma^2 y^2}, \qquad \gamma^2 = \frac{4M^2 x^2}{Q^2}; \qquad \varepsilon = \frac{2(1-y)}{1 + (1-y)^2} \quad (M = 0). \tag{1.1.26}$$

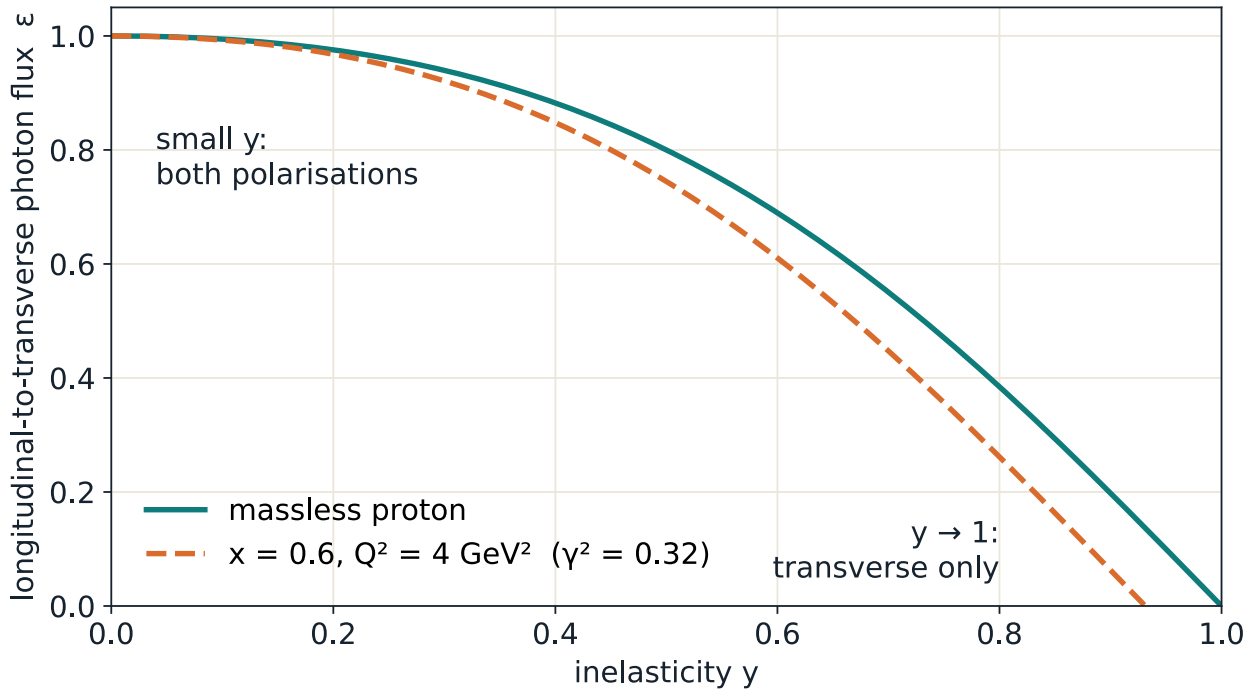


Figure 9. The ratio ε of longitudinal to transverse photon flux, Eq. (1.1.26). The dashed curve keeps the proton mass at a point typical of low-energy fixed-target data, where $\gamma^2 = 0.32$ and the largest allowed y is 0.93.

The role of y

Figure 9 shows the content of Eq. (1.1.26). At small y the lepton is barely deflected, $\varepsilon \to 1$, and it radiates both polarisations equally. At the largest y, $\varepsilon \to 0$ and only transverse photons are exchanged.

The second limit is the right-hand panel of Figure 8. Take $M = 0$. At $y = 1$, $P \cdot k' = 0$: in the centre-of-mass frame the lepton is scattered straight back, and the photon's momentum lies along the collision axis. A massless lepton keeps its helicity at the photon vertex, so its spin component along the axis changes by one unit. The photon must carry that unit, which a longitudinal photon, with spin component zero, cannot do.

So y controls the polarisation of the probe. Two consequences follow.

- **Measuring** R requires the same (x, Q^2) at different ε, that is, at different beam energies. Since ε differs appreciably from 1 only at large y, F_L is hard to measure and is known much less precisely than F_2.
- R **tests the spin of the charged constituents.** Chapter 1.2 shows that a spin-$\frac{1}{2}$ constituent cannot absorb a longitudinal photon, giving $R \to 0$, while a spin-0 constituent cannot absorb a transverse one, giving $F_1 = 0$. The data show $F_L \ll F_2$ [1].

In hand: a meaning for each function. F_1 is the response to transverse photons, F_L to longitudinal ones, and y sets the mixture.

9. Elastic scattering, resonances and the deep inelastic region

The same formula covers three very different outcomes, told apart by the mass W of what comes out. Figure 10 lays them along one axis. If the proton survives intact, $W = M$: a single sharp line. If it is excited to a short-lived state, there is a bump at that state's mass. If it breaks up, W can take any value, and the bumps merge into a continuum.

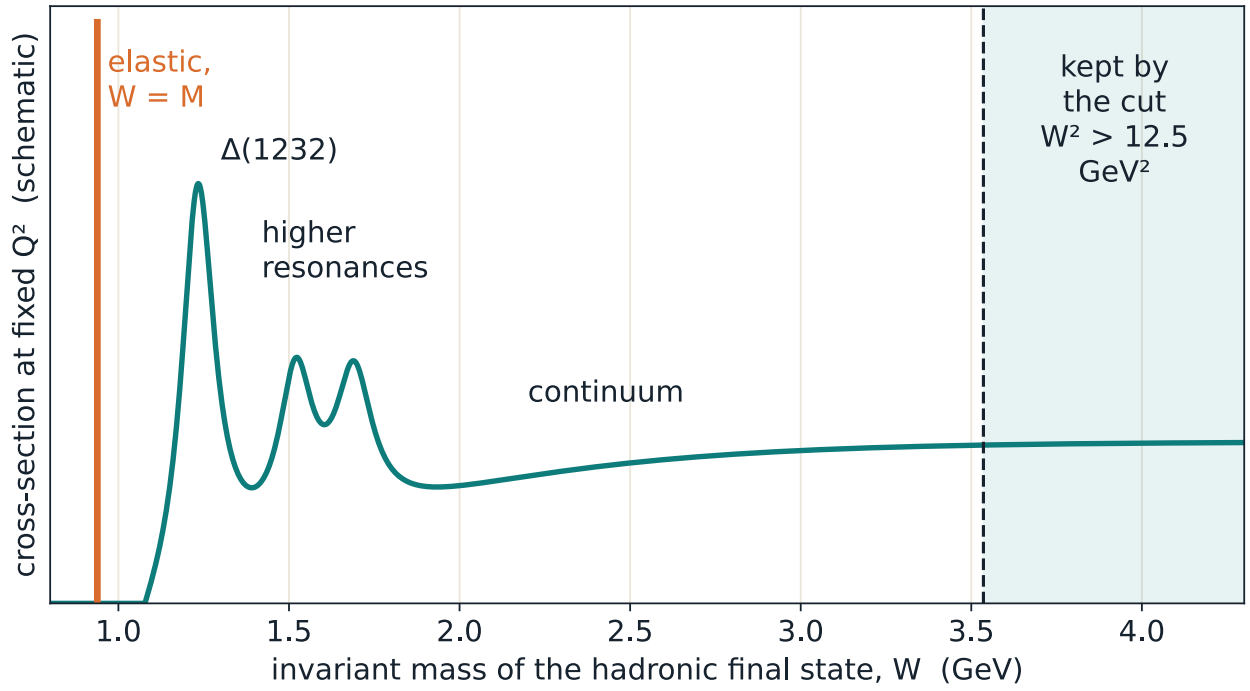


Figure 10. How the inclusive cross-section at a fixed, moderate Q^2 depends on the hadronic mass W. This is a sketch from a toy formula, not data: the positions of the features are real, their heights are not. The elastic peak and the resonances fall rapidly with Q^2. The continuum does not.

The elastic limit

In elastic scattering X is a single proton. Then $W^2 = M^2$, and the hadronic tensor is proportional to $\delta(1-x)$. The matrix element of the current between two proton states is described by two form factors, the electric $G_E(Q^2)$ and the magnetic $G_M(Q^2)$, normalised so that $G_E(0) = 1$ and $G_M(0) = 2.79$, the proton's charge and magnetic moment. The structure functions are (derivation D7)

$$\begin{aligned} F_1^{\text{el}} &= \tfrac{1}{2}\, G_M^2(Q^2)\; \delta(1-x), \\ F_2^{\text{el}} &= \frac{G_E^2(Q^2) + \tau\, G_M^2(Q^2)}{1+\tau}\; \delta(1-x), \qquad \tau = \frac{Q^2}{4M^2}. \end{aligned} \tag{1.1.27}$$

Integrating Eq. (1.1.22) over E' across the peak gives the Rosenbluth formula [5]:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \cos^2\frac{\theta}{2}}{4E^2 \sin^4\frac{\theta}{2}}\, \frac{E'}{E} \left[\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau\, G_M^2 \tan^2\frac{\theta}{2} \right]. \tag{1.1.28}$$

Two cases matter.

- **A point particle.** For a structureless spin-$\frac{1}{2}$ particle of unit charge, $G_E = G_M = 1$ at all Q^2. Then $F_2^{\text{el}} = \delta(1 - x)$ and $2xF_1^{\text{el}} = F_2^{\text{el}}$, with no dependence on Q^2. This is the lowest-order quark structure function of chapter 1.4, Eq. (1.4.5).
- **The proton.** The measured form factors follow the dipole shape $G_E \approx G_M/2.79 \approx (1 + Q^2/0.71\ \text{GeV}^2)^{-2}$ to a fair approximation. At $Q^2 = 1$ GeV^2 this gives $G_E^2 = 0.030$, and at 10 GeV^2 it gives 2×10^{-5}. The elastic structure functions fall as Q^{-8}. An extended object rarely holds together when struck hard.

Resonances

Just above the elastic peak, starting at the threshold for producing one pion, $W = M + m_\pi = 1.08$ GeV, the cross-section shows a series of bumps. Each is an excited state of the nucleon. The most prominent is the $\Delta(1232)$, and two further groups lie near 1.5 and 1.7 GeV. Each shrinks rapidly as Q^2 grows, because exciting a specific state requires the whole proton to respond coherently.

The deep inelastic region

Above about $W = 2$ GeV the bumps merge into a smooth continuum in which many hadrons are produced. The deep inelastic region is defined by $Q^2 \gg M^2$ and $W^2 \gg M^2$, so that the photon resolves distances much smaller than the proton and the final state is far from any single resonance. By Eq. (1.1.4), at fixed Q^2 the resonance region is the region of large x: the shaded strip along the right-hand edge of Figure 3.

Fits of parton distributions keep only data in this region, by cuts on both variables. Ours require $Q^2 > 3.49$ GeV^2 and $W^2 > 12.5$ GeV^2 (Phase B ledger), that is $W > 3.5$ GeV. These are the two lines in Figure 5. At the lowest allowed Q^2 the cut on W^2 removes everything above $x = 0.23$, and at higher Q^2 the limit moves towards $x = 1$.

In hand: elastic scattering and the resonances as special cases of the same two functions, and the cuts that define the region the rest of the guide uses.

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10. Weak currents and the third structure function

The photon is not the only particle a lepton can exchange with the proton. The Z and W bosons do the same job, with two differences. Their interactions distinguish left from right, which lets a third structure function appear. And the W changes the flavour of the quark that absorbs it, so each sign of W is absorbed by only some of the quarks. Figure 11 puts the three side by side. Results are stated without full derivation; the parton-model content belongs to chapter 1.2.

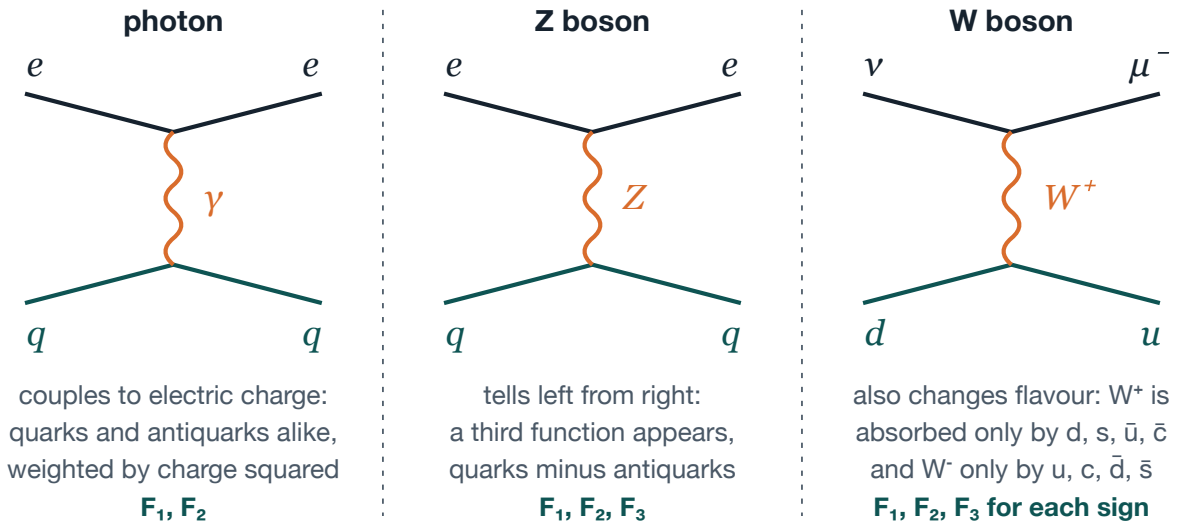


Figure 11. The three exchanges and what each can tell about the quarks in the proton. The flavour statements are parton-model results quoted from ESW, section 4.1 [1], and derived in chapter 1.2.

What changes

The weak currents are not pure vectors. They have the form $V - A$, vector minus axial vector, and they do not conserve parity. The term excluded by parity in section 6 is now allowed. Lepton masses being negligible, terms in $W^{\alpha\beta}$ proportional to q^α or q^β still drop out of the cross-section, so exactly one new function matters:

$$W^{\alpha\beta} = -g^{\alpha\beta} F_1 + \frac{P^\alpha P^\beta}{\nu} F_2 - i\,\epsilon^{\alpha\beta\rho\sigma} \frac{P_\rho\, q_\sigma}{2\nu} F_3 + \big(\text{terms with } q^\alpha \text{ or } q^\beta\big), \tag{1.1.29}$$

with $\epsilon_{0123} = +1$. The new term, the last row of the table in section 6, is antisymmetric. It contributes because the leptonic tensor for a left-handed neutrino also has an antisymmetric part. F_3 is the interference between the vector and axial parts of the current.

Charged current: neutrino scattering

A neutrino beam on a nucleon exchanges a W boson, $\nu p \to \ell^- X$. With G_F the Fermi constant and M_W the mass of the W boson, the cross-section is

$$\frac{d^2\sigma^{\nu,\bar\nu}}{dx\,dy} = \frac{G_F^2\,(s - M^2)}{2\pi} \left(\frac{M_W^2}{Q^2 + M_W^2}\right)^2 \left[x\,y^2 F_1 + \left(1 - y - \frac{M^2 x^2 y^2}{Q^2}\right) F_2 \pm y\left(1 - \frac{y}{2}\right) x F_3 \right], \tag{1.1.30}$$

with the upper sign for neutrinos and the lower for antineutrinos. Each beam has its own set of structure functions. For $Q^2 \ll M_W^2$ and a fixed target this is ESW's Eq. (4.3). The first two terms carry the weights of Figure 7, since the symmetric part of the neutrino's leptonic tensor is four times Eq. (1.1.11); the script verifies the whole bracket with explicit Dirac matrices. The prefactor has no $1/Q^4$. The neutrino cross-section does not fall with Q^2 until Q^2 approaches M_W^2, and it grows in proportion to s.

To see what F_3 adds, set $M = 0$ and $2xF_1 = F_2$. The bracket becomes

$$\left(1-y+\frac{y^2}{2}\right)F_2 \pm y\left(1-\frac{y}{2}\right)xF_3 = \tfrac{1}{2}\left(F_2 \pm xF_3\right) + \tfrac{1}{2}\,(1-y)^2\,\left(F_2 \mp xF_3\right). \tag{1.1.31}$$

The two combinations $F_2 \pm xF_3$ come with different dependences on y, one flat and one falling as $(1-y)^2$, so the distribution in y separates them. In the parton model (ESW, section 4.1 [1]; chapter 1.2) the neutrino's W^+ is absorbed only by d, s, $\bar{u}$ and $\bar{c}$, which gives $F_2^\nu = 2x\,[d+s+\bar{u}+\bar{c}]$ and $xF_3^\nu = 2x\,[d+s-\bar{u}-\bar{c}]$. So $F_2 + xF_3$ counts quarks and $F_2 - xF_3$ counts antiquarks. The photon cannot make that distinction: its F_2 is a sum over quarks and antiquarks of every flavour, weighted by e_q^2.

Neutral current and charged current with electrons

At HERA, Q^2 reaches M_Z^2 and M_W^2, and the electron itself probes the proton through the weak interaction. For neutral-current scattering, $e^\mp p \to e^\mp X$, photon exchange, Z exchange and their interference add up to (ESW Eq. (4.19) [1])

$$\frac{d^2\sigma^{\mathrm{NC}}_{e^\mp p}}{dx\,dQ^2} = \frac{4\pi\alpha^2}{x\,Q^4}\left[x\,y^2F_1^{\mathrm{NC}} + (1-y)\,F_2^{\mathrm{NC}} \pm y\left(1-\frac{y}{2}\right)xF_3^{\mathrm{NC}}\right] \qquad (M=0). \tag{1.1.32}$$

The generalised structure functions reduce to the electromagnetic ones for $Q^2 \ll M_Z^2$, where $F_3^{\mathrm{NC}} \to 0$. The sign of the last term is opposite for electrons and positrons, so the difference of the two cross-sections isolates xF_3^{NC}, which again measures quarks minus antiquarks.

Charged-current scattering, $e^- p \to \nu X$ and $e^+ p \to \bar{\nu} X$, has the form of Eq. (1.1.30) with an extra factor $\frac{1}{2}$ for an unpolarised beam, because only one of the electron's two helicity states couples to the W. An e^- emits a W^-, which is absorbed by u-type quarks and d-type antiquarks. An e^+ emits a W^+ and probes the opposite set.

In hand: a third structure function, and several new combinations of quark distributions: neutrino and high-Q^2 data tell quarks from antiquarks, which photon exchange cannot.

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11. Scaling, and where this enters our fits

An object with a size responds less and less as the wavelength of the probe shrinks below that size. That is why the elastic form factors collapse: their scale is the inverse size of the proton, and they fall as a power of $Q^2/0.71\ \mathrm{GeV}^2$. The inelastic structure functions do not: at fixed x they stay nearly the same as Q^2 is raised. Figure 12 sets the two behaviours side by side.

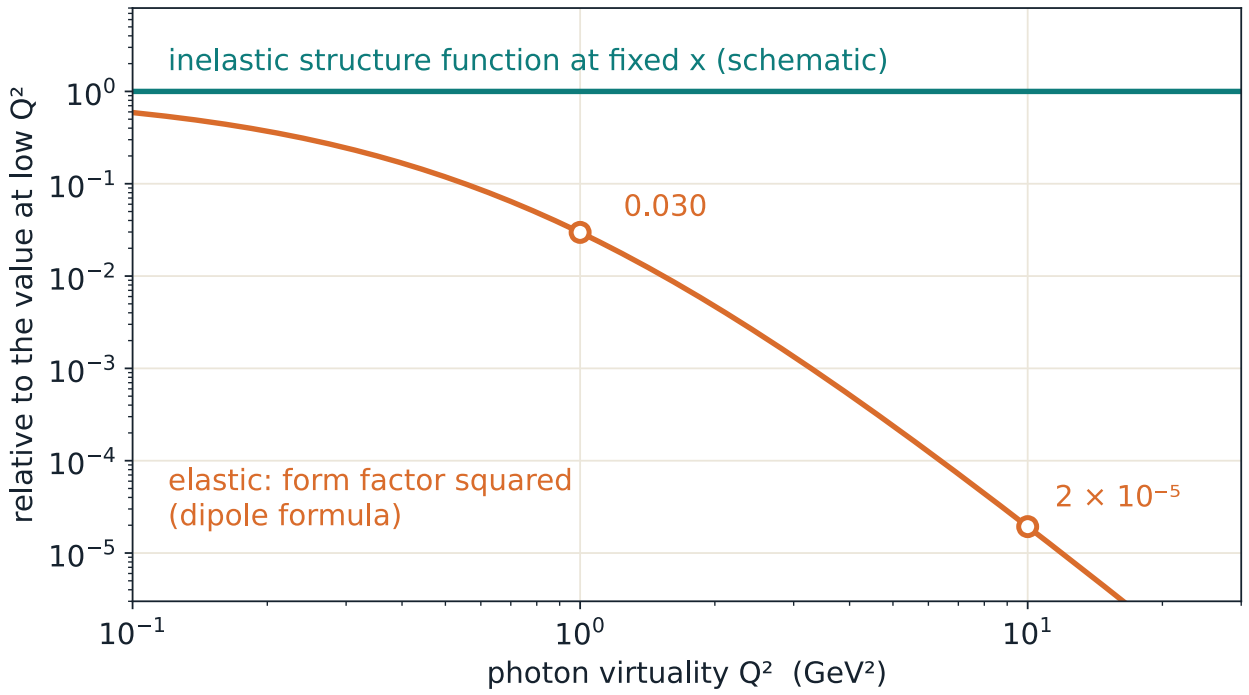


Figure 12. Elastic against inelastic as the magnification is raised. The falling curve is the square of the dipole form factor of section 9, with the two values quoted there marked. The flat line is schematic: it stands for Eq. (1.1.33) and is not data.

The experimental fact

The structure functions are dimensionless, but they could depend on Q^2 through its ratio to any fixed scale of the target. In the limit $Q^2 \to \infty$ and $\nu \to \infty$ with x fixed, called the Bjorken limit, they are observed to become functions of x alone:

$$F_i(x, Q^2) \;\longrightarrow\; F_i(x). \tag{1.1.33}$$

This is *Bjorken scaling* [3]. It was first seen at SLAC, in experiments with electron beams of up to 20 GeV, at Q^2 of a few GeV^2 [4]. ESW's Figure 4.2 collects F_2 measurements on protons from the SLAC–MIT, BCDMS, H1 and ZEUS experiments, with Q^2 from 1.5 to 1000 GeV^2. Over those three orders of magnitude the points lie close to one curve in x [1].

No length scale appears in the inelastic structure functions. Whatever absorbs the photon looks the same at every magnification, as a point does. Chapter 1.2 turns this into the parton model. The scaling is approximate: F_2 falls slowly with Q^2 at large x and rises at small x [1]. Chapter 1.4 derives that pattern.

In this project

Three things in our fits are set by this chapter.

- **The data are points on the plane of Figure 5.** The first Phase A fits used four deep inelastic datasets, the SLAC and BCDMS measurements of F_2 on protons and deuterons: 648 points after cuts (33, 34, 333 and 248). The full deep inelastic set has 19 datasets, from SLAC, BCDMS, NMC, HERA (neutral current, charged current and heavy flavour), CHORUS and NuTeV (Results ledger).
- **The cuts and the mass terms.** The Phase B fits apply $Q^2 > 3.49\ \mathrm{GeV}^2$ and $W^2 > 12.5\ \mathrm{GeV}^2$ and have target-mass corrections switched on (Phase B ledger). Section 9 is the reason for the cuts and section 7 the origin of the

mass terms.

- **Neutrino data carry flavour information that photon data cannot.** CHORUS and NuTeV are neutrino experiments, described by Eq. (1.1.30). In the leave-one-out Fisher test of Phase B, removing those four datasets from the global set raised the number of unconstrained parameter directions from 8 to 9 of 52, which is 0.250 per dataset, against 0.133 per dataset for the fifteen charged-lepton sets (Phase B ledger, PPDF-38).

In hand: the language of the measurement, two functions of x and Q^2 defined without any model, and the observation that chapter 1.2 explains.

Derivations

D1. The kinematic relations and ranges

Derives (1.1.1), (1.1.4), (1.1.5), (1.1.6), (1.1.7), (1.1.8), (1.1.9)

The virtuality. Neglect the lepton's mass, so that $k^2 = k'^2 = 0$. Then $q^2 = (k - k')^2 = -2\,k \cdot k'$. With the energies and the deflection angle taken in any one frame, $k \cdot k' = EE'(1 - \cos\theta) = 2EE' \sin^2(\theta/2)$. This is Eq. (1.1.1).

The relation between the invariants. From the definitions (1.1.3), $x\,y = Q^2/(2P \cdot k)$. Since $k^2 = 0$, $s = (k + P)^2 = M^2 + 2P \cdot k$. Hence $Q^2 = 2\,(P \cdot k)\,x\,y = x\,y\,(s - M^2)$, which is Eq. (1.1.5).

The hadronic mass. $W^2 = (P + q)^2 = M^2 + 2P \cdot q + q^2 = M^2 + 2\nu - Q^2$. With $2\nu = Q^2/x$ this is $M^2 + Q^2(1 - x)/x$, Eq. (1.1.4).

The range of x. $Q^2 > 0$ by Eq. (1.1.1), and $\nu = M(E - E') > 0$ in the proton's rest frame because the proton, initially at rest, can only gain energy. So $x > 0$. The lightest state with the proton's baryon number is the proton, so $W^2 \geq M^2$, and by Eq. (1.1.4) $Q^2(1 - x)/x \geq 0$, that is $x \leq 1$.

The range of y. In the rest frame $y = 1 - E'/E$, which lies between 0 and 1. The upper end is tighter when M is kept. Equation (1.1.1) with $\sin^2(\theta/2) \leq 1$ gives $Q^2 \leq 4EE'$. Substitute $Q^2 = 2MExy$ and $E' = E(1 - y)$:

$$2MExy \leq 4E^2(1 - y) \quad \Longrightarrow \quad y \leq \frac{1}{1 + Mx/2E} = \frac{1}{1 + M^2x/(s - M^2)},$$

using $s - M^2 = 2ME$. This is Eq. (1.1.6). The limit is reached for backward scattering, $\theta = \pi$.

The Breit frame. In the proton's rest frame $q^\mu = (q^0, \mathbf{q})$ with $q^0 = E - E'$ and $|\mathbf{q}|^2 = (q^0)^2 + Q^2 > (q^0)^2$. A boost along $\mathbf{q}$ with velocity $\beta = q^0/|\mathbf{q}| < 1$ gives $q'^0 = \gamma_\beta\,(q^0 - \beta|\mathbf{q}|) = 0$, where $\gamma_\beta = (1 - \beta^2)^{-1/2}$. In that frame $q'^\mu = (0, \mathbf{q}')$ with $|\mathbf{q}'| = Q$. Such a boost exists only because q is space-like.

The light-cone decomposition. In Eq. (1.1.7), $p^2 = n^2 = 0$, $n \cdot p = 1$, and q_T is orthogonal to both p and n. Then $P^2 = 2 \cdot \frac{1}{2}M^2\,n \cdot p = M^2$, $q^2 = -q_T^2 = -Q^2$, $n \cdot q = 0$ and $P \cdot q = \nu\,p \cdot n = \nu$.

Fixed target. Put $P = (M, \mathbf{0})$ into Eq. (1.1.3): $\nu = M(E - E')$ and $P \cdot k = ME$. With Eq. (1.1.1) this gives Eq. (1.1.8).

Collider. Neglect M against E_p. Take the proton along $+z$ and the lepton along $-z$, and let the lepton be deflected by θ:

$$P = E_p(1, 0, 0, 1), \qquad k = E_e(1, 0, 0, -1), \qquad k' = E'(1, \sin\theta, 0, -\cos\theta).$$

Then $P\cdot k = 2E_eE_p$, so $s = 4E_eE_p$, and $P\cdot k' = E_pE'(1+\cos\theta) = 2E_pE'\cos^2(\theta/2)$. Therefore

$$y = \frac{P\cdot k - P\cdot k'}{P\cdot k} = 1 - \frac{E'}{E_e}\cos^2\frac{\theta}{2}.$$

Q^2 is Eq. (1.1.1) with $E = E_e$, and x follows from Eq. (1.1.5). This is Eq. (1.1.9).

D2. The leptonic tensor

Derives (1.1.11)

Squaring the lepton current. The lepton factor in Eq. (1.1.10) is $\bar u(k')\gamma^\alpha u(k)$. Its complex conjugate is $\bar u(k)\gamma^\beta u(k')$, because $(\gamma^\beta)^\dagger = \gamma^0\gamma^\beta\gamma^0$. Average over the two spin states of the incoming lepton and sum over those of the outgoing one:

$$L^{\alpha\beta} = \frac{1}{2}\sum_{\text{spins}} \bar u(k')\gamma^\alpha u(k)\;\bar u(k)\gamma^\beta u(k').$$

The trace. For a massless spinor $\sum_{\text{spins}} u(k)\bar u(k) = \not{k}$. Using it twice turns the product into a trace over Dirac indices:

$$L^{\alpha\beta} = \frac{1}{2}\,\mathrm{Tr}\big[\not{k}'\gamma^\alpha\not{k}\gamma^\beta\big] = \frac{1}{2}\,k'_\rho k_\sigma\,\mathrm{Tr}\big[\gamma^\rho\gamma^\alpha\gamma^\sigma\gamma^\beta\big].$$

Evaluating it. The trace of four gamma matrices is $\mathrm{Tr}[\gamma^\rho\gamma^\alpha\gamma^\sigma\gamma^\beta] = 4\,(g^{\rho\alpha}g^{\sigma\beta} - g^{\rho\sigma}g^{\alpha\beta} + g^{\rho\beta}g^{\alpha\sigma})$. Contracting with $k'_\rho k_\sigma$:

$$L^{\alpha\beta} = 2\big(k'^\alpha k^\beta - g^{\alpha\beta}\,k\cdot k' + k'^\beta k^\alpha\big),$$

which is Eq. (1.1.11).

Current conservation. With $q = k - k'$ and $k^2 = k'^2 = 0$: $q\cdot k = -k\cdot k'$ and $q\cdot k' = k\cdot k'$. Then

$$q_\alpha L^{\alpha\beta} = 2\big[(q\cdot k)\,k'^\beta + (q\cdot k')\,k^\beta - q^\beta\,k\cdot k'\big] = 2\,k\cdot k'\,\big[-k'^\beta + k^\beta - q^\beta\big] = 0.$$

Two contractions used later. $g_{\alpha\beta}L^{\alpha\beta} = 2\,(2-4)\,k\cdot k' = -4\,k\cdot k' = -2Q^2$, and $L^{\alpha\beta}P_\alpha P_\beta = 4\,(P\cdot k)(P\cdot k') - 2M^2\,k\cdot k' = 4\,(P\cdot k)(P\cdot k') - M^2Q^2$.

D3. The hadronic tensor as a current correlator

Derives (1.1.13)

Translating the current. The momentum operator $\hat p$ generates translations: $j(z) = e^{i\hat p\cdot z}\,j(0)\,e^{-i\hat p\cdot z}$. Between momentum eigenstates,

$$\langle P|\,j^\dagger_\beta(z)\,|X\rangle = e^{i(P-p_X)\cdot z}\,\langle P|\,j^\dagger_\beta(0)\,|X\rangle.$$

The delta function as an integral. Write $(2\pi)^4\delta^4(P+q-p_X) = \int d^4z\; e^{i(P+q-p_X)\cdot z}$ in Eq. (1.1.12). The factor $e^{i(P-p_X)\cdot z}$ is exactly what the previous step absorbs:

$$W_{\alpha\beta} = \frac{1}{4\pi} \int d^4z \, e^{iq\cdot z} \sum_X \langle P| \, j_\beta^\dagger(z) \, |X\rangle\langle X| \, j_\alpha(0) \, |P\rangle .$$

Completeness. The sum over X in Eq. (1.1.12) runs over all states with their phase space, with no restriction, since the detector does not select among them. So $\sum_X |X\rangle\langle X| = 1$, and Eq. (1.1.13) follows. An inclusive measurement is what makes this step possible.

The commutator. The product can be replaced by the commutator $[\, j_\beta^\dagger(z), j_\alpha(0)\,]$, the form used in the operator analysis. The extra term, with the currents in the opposite order, would need intermediate states of momentum $p_X = P - q$. In the proton's rest frame such a state has energy $M - q^0 < M$ and baryon number one. No such state exists, so the extra term vanishes.

D4. Why there are exactly two structure functions

Derives (1.1.15), (1.1.16)

The general form. A two-index tensor built from P, q, $g^{\alpha\beta}$ and $\epsilon^{\alpha\beta\rho\sigma}$ has six possible terms:

$$\begin{aligned} W^{\alpha\beta} = &- W_1 \, g^{\alpha\beta} + W_2 \, P^\alpha P^\beta + W_4 \, q^\alpha q^\beta + W_5 \left(P^\alpha q^\beta + q^\alpha P^\beta\right) \\ &+ i \, W_6 \left(P^\alpha q^\beta - q^\alpha P^\beta\right) + i \, W_3 \, \epsilon^{\alpha\beta\rho\sigma} P_\rho q_\sigma . \end{aligned}$$

Each W_i is a function of ν and Q^2. No other vector is available once the proton's spin has been averaged over.

Hermiticity. Taking the complex conjugate of Eq. (1.1.12) exchanges the two matrix elements, so $W^*_{\alpha\beta} = W_{\beta\alpha}$. The symmetric terms then have real coefficients, and the factors of i in front of the antisymmetric terms make W_6 and W_3 real as well.

Parity. Under a reflection of space, a vector current and the vectors P and q all transform in the same way, and the strong interaction is invariant. $W^{\alpha\beta}$ must then be a true tensor. The last term is a pseudotensor, because $\epsilon^{\alpha\beta\rho\sigma}$ changes sign under reflection. So $W_3 = 0$ for photon exchange.

Current conservation. Contract with q_α and collect the coefficients of the two independent vectors P^β and q^β, using $q \cdot P = \nu$ and $q^2 = -Q^2$:

$$q_\alpha W^{\alpha\beta} = \left[\nu W_2 - Q^2 W_5 + i Q^2 W_6\right] P^\beta + \left[\, - W_1 - Q^2 W_4 + \nu W_5 + i \nu W_6\right] q^\beta = 0 .$$

Contracting with q_β instead gives the same two brackets with $W_6 \to -W_6$. Both versions of the first bracket vanish only if

$$W_6 = 0, \qquad W_5 = \frac{\nu}{Q^2} \, W_2, \qquad \text{and then} \qquad W_4 = \frac{\nu^2}{Q^4} \, W_2 - \frac{1}{Q^2} \, W_1 .$$

Three coefficients are fixed and two, W_1 and W_2, are free. The check script confirms the count by computing the rank of these eight linear conditions on five coefficients.

Reassembling. Substitute back. The terms with W_1 combine to $W_1 \, (-g^{\alpha\beta} - q^\alpha q^\beta / Q^2) = W_1 \, (-g^{\alpha\beta} + q^\alpha q^\beta / q^2)$. The terms with W_2 form a perfect square:

$$W_2 \Big[P^\alpha P^\beta + \frac{\nu}{Q^2} \left(P^\alpha q^\beta + q^\alpha P^\beta\right) + \frac{\nu^2}{Q^4} \, q^\alpha q^\beta \Big] = W_2 \Big(P^\alpha + \frac{\nu}{Q^2} q^\alpha \Big) \Big(P^\beta + \frac{\nu}{Q^2} q^\beta \Big) .$$

Since $\nu/Q^2 = 1/2x$, the vector in brackets is $\hat{P}$. Setting $F_1 = W_1$ and $F_2 = \nu W_2$ gives Eq. (1.1.15).

A remark on the source. ESW's Eq. (4.29) prints the first bracket with the opposite overall sign. The sign in Eq. (1.1.15) is the one consistent with ESW's own projections, their Eq. (4.35), with Moore's Eq. (1.18) and with a positive F_1; the check script confirms it.

The projections. From Eq. (1.1.7), $n \cdot q = 0$, $n^2 = 0$ and $n \cdot \hat{P} = n \cdot P = 1$. So $n_\alpha n_\beta W^{\alpha\beta} = F_2/\nu$, with no approximation. For $M = 0$, $P = p$ with $p^2 = 0$ and $p \cdot q = \nu$. Then $p_\alpha p_\beta \left(-g^{\alpha\beta} + q^\alpha q^\beta/q^2\right) = -\nu^2/Q^2 = -\nu/2x$ and $p \cdot \hat{P} = \nu/2x$, so

$$p_\alpha p_\beta W^{\alpha\beta} = -\frac{\nu}{2x} F_1 + \frac{\nu}{4x^2} F_2 = \frac{\nu}{4x^2}\left(F_2 - 2x F_1\right),$$

which is Eq. (1.1.16).

D5. The contraction and the differential cross-section

Derives (1.1.14), (1.1.18), (1.1.19), (1.1.20), (1.1.21), (1.1.22)

The cross-section. For a massless lepton on a proton the flux factor is $4\,P \cdot k = 2(s - M^2)$. The spin-averaged squared amplitude from Eq. (1.1.10), summed over X with its phase space, is $(e^4/Q^4)\, L^{\alpha\beta} \cdot 4\pi\, W_{\alpha\beta}$ by the definitions (1.1.11) and (1.1.12). The remaining phase space is that of the scattered lepton:

$$d\sigma = \frac{1}{2(s - M^2)} \, \frac{e^4}{Q^4} \, 4\pi \, L^{\alpha\beta} W_{\alpha\beta} \, \frac{d^3k'}{(2\pi)^3 \, 2E'} = \frac{2\alpha^2}{(s - M^2)\, Q^4} \, L^{\alpha\beta} W_{\alpha\beta} \, \frac{d^3k'}{E'},$$

using $e^4 = 16\pi^2\alpha^2$.

The Jacobian. d^3k'/E' is Lorentz invariant, so it can be evaluated in the proton's rest frame, where $d^3k'/E' = E'\, dE'\, d\cos\theta\, d\phi$. From Eq. (1.1.8), $y = 1 - E'/E$, so $|dE'| = E\, dy$. At fixed E', $x = 2EE'(1 - \cos\theta)/(2MEy)$, so $|d\cos\theta| = (My/E')\, dx$. Therefore

$$\frac{d^3k'}{E'} = M\, E\, y\; dx\, dy\, d\phi = \frac{s - M^2}{2}\; y\; dx\, dy\, d\phi.$$

Nothing depends on the azimuthal angle ϕ for an unpolarised target, so its integral gives 2π. The factors of $s - M^2$ cancel and Eq. (1.1.14) follows.

The scalar products. From Eq. (1.1.5) and the definition of y:

$$P \cdot k = \frac{s - M^2}{2} = \frac{Q^2}{2xy}, \qquad P \cdot k' = P \cdot k - P \cdot q = \frac{Q^2}{2xy}\,(1 - y).$$

The contraction. Terms in Eq. (1.1.15) with a factor q^α or q^β vanish against $L^{\alpha\beta}$ (derivation D2). With the two contractions at the end of D2 and $1/\nu = 2x/Q^2$:

$$\begin{aligned} L^{\alpha\beta} W_{\alpha\beta} &= 2Q^2 F_1 + \frac{2x}{Q^2}\left[\frac{Q^4}{x^2y^2}\,(1 - y) - M^2Q^2\right] F_2 \\ &= \frac{2Q^2}{x\,y^2}\left[x\,y^2 F_1 + \left(1 - y - \frac{M^2x^2y^2}{Q^2}\right) F_2\right], \end{aligned}$$

which is Eq. (1.1.18). Multiplying by $2\pi\alpha^2 y/Q^4$ gives Eq. (1.1.19), and the factor y/Q^2 from $dy = (y/Q^2)\,dQ^2$ gives Eq. (1.1.20).

The form with F_L. For $M = 0$, $2xF_1 = F_2 - F_L$, so $x\,y^2 F_1 + (1-y)F_2 = \tfrac{1}{2}\big[(y^2+2-2y)F_2 - y^2 F_L\big] = \tfrac{1}{2}\big[Y_+ F_2 - y^2 F_L\big]$. This is Eq. (1.1.21). With the mass kept and F_L in its general form, the same steps give the bracket $\big(Y_+ + \tfrac{1}{2}\gamma^2 y^2\big)F_2 - y^2 F_L$, with $\gamma^2 = 4M^2x^2/Q^2$.

The mass term in the rest frame. By Eq. (1.1.8), $M^2x^2y^2/Q^2 = Mxy/2E = Q^2/4E^2 = (E'/E)\sin^2(\theta/2)$, and $1 - y = E'/E$. So $1 - y - M^2x^2y^2/Q^2 = (E'/E)\cos^2(\theta/2)$.

The laboratory form. From the Jacobian, $dx\,dy = E'\,dE'\,d\Omega/(2\pi MEy)$ after the ϕ integral is restored into $d\Omega$. So $d^2\sigma/dE'd\Omega = (E'/2\pi MEy)\,d^2\sigma/dx\,dy$. Insert Eq. (1.1.19) and use $Q^2 = 4EE'\sin^2(\theta/2)$, $xy = Q^2/2ME$ and $y = (E-E')/E$. The F_1 term becomes $(2\alpha^2E'/MEQ^2)F_1 = \alpha^2\cdot 2\sin^2(\theta/2)\,F_1/\big(4E^2\sin^4(\theta/2)\,M\big)$, and the F_2 term becomes $\alpha^2\cos^2(\theta/2)\,F_2/\big(4E^2\sin^4(\theta/2)\,(E-E')\big)$. Their sum is Eq. (1.1.22).

D6. Photon polarisation and the parameter ε

Derives (1.1.17), (1.1.23), (1.1.24), (1.1.25), (1.1.26)

The polarisation vectors. The vectors orthogonal to a space-like q span a three-dimensional space with one time-like and two space-like directions. In the proton's rest frame with $\mathbf{q}$ along z, $q^\mu = (q^0, 0, 0, |\mathbf{q}|)$ and

$$\varepsilon_\pm^\mu = \frac{1}{\sqrt{2}}\,(0, 1, \pm i, 0), \qquad \varepsilon_L^\mu = \frac{1}{Q}\,(|\mathbf{q}|, 0, 0, q^0).$$

These obey $\varepsilon\cdot q = 0$, $\varepsilon_\pm\cdot\varepsilon_\pm^* = -1$ and $\varepsilon_L^2 = (|\mathbf{q}|^2 - (q^0)^2)/Q^2 = +1$. The tensor $g^{\alpha\beta} - q^\alpha q^\beta/q^2$ is the projector onto this space, so with the signs of the norms

$$-g^{\alpha\beta} + \frac{q^\alpha q^\beta}{q^2} = \sum_{\lambda=\pm}\varepsilon_\lambda^\alpha\varepsilon_\lambda^{\beta *} - \varepsilon_L^\alpha\varepsilon_L^\beta.$$

The longitudinal vector covariantly. $\hat{P}$ is orthogonal to q, and in the rest frame it has no transverse components, so it is proportional to ε_L. Its norm is

$$\hat{P}^2 = M^2 + \frac{\nu}{x} - \frac{Q^2}{4x^2} = M^2 + \frac{Q^2}{4x^2} = \frac{Q^2}{4x^2}\,(1+\gamma^2),$$

using $\nu/x = Q^2/2x^2$. Hence $\varepsilon_L = \hat{P}/\sqrt{\hat{P}^2}$.

The hadronic tensor. Insert both results into Eq. (1.1.15):

$$W^{\alpha\beta} = F_1\sum_{\lambda=\pm}\varepsilon_\lambda^\alpha\varepsilon_\lambda^{\beta *} + \Big[-F_1 + \frac{\hat{P}^2}{\nu}F_2\Big]\varepsilon_L^\alpha\varepsilon_L^\beta.$$

With $\hat{P}^2/\nu = (1+\gamma^2)/2x$, the bracket is $\big[(1+\gamma^2)F_2 - 2xF_1\big]/2x = F_L/2x$. This is Eq. (1.1.23), and it is the reason the general F_L in Eq. (1.1.17) carries the factor $1 + 4M^2x^2/Q^2$.

The leptonic tensor. Define $L_L = L^{\alpha\beta}\varepsilon_{L\alpha}\varepsilon_{L\beta}$ and $L_T = \sum_\pm L^{\alpha\beta}\varepsilon_{\lambda\alpha}^*\varepsilon_{\lambda\beta}$. Because $q_\alpha L^{\alpha\beta} = 0$, $\hat{P}$ may be replaced by P, and the contractions of D2 and the scalar products of D5 give

$$L_L = \frac{4(P\cdot k)(P\cdot k') - M^2Q^2}{\hat{P}^2} = \frac{4Q^2}{y^2}\,\frac{1 - y - \frac{1}{4}\gamma^2 y^2}{1+\gamma^2}.$$

The completeness relation gives $L_T - L_L = -g_{\alpha\beta}L^{\alpha\beta} = 2Q^2$, so

$$L_T = \frac{4Q^2}{y^2}\,\frac{1 - y + \frac{1}{2}y^2 + \frac{1}{4}\gamma^2 y^2}{1+\gamma^2}.$$

The ratio $\varepsilon = L_L/L_T$ is Eq. (1.1.26).

The cross-section. By Eq. (1.1.23), $L^{\alpha\beta}W_{\alpha\beta} = L_T F_1 + L_L\,F_L/2x = L_T\left[F_1 + \varepsilon\,F_L/2x\right]$. Insert this into Eq. (1.1.14) with the factor y/Q^2 for dQ^2, and express F_1 and F_L through Eq. (1.1.24):

$$\frac{d^2\sigma}{dx\,dQ^2} = \frac{2\pi\alpha^2 y^2}{Q^6}\,L_T\,\frac{\nu(1-x)}{4\pi^2\alpha}\left[\sigma_T + \varepsilon\,\sigma_L\right].$$

From the two expressions above, $1-\varepsilon = 2Q^2/L_T$, so $L_T = 2Q^2/(1-\varepsilon)$. With $\nu = Q^2/2x$ the prefactor is $\alpha(1-x)y^2/\big(2\pi x Q^2(1-\varepsilon)\big)$, which is Γ in Eq. (1.1.25).

The photon cross-section. For completeness, the origin of Eq. (1.1.24). The cross-section for a photon of polarisation ε_λ on a proton is the squared amplitude $e^2|\varepsilon_\lambda^\alpha\langle X|j_\alpha|P\rangle|^2$, summed over X with its phase space and divided by a flux factor $4MK$. By Eq. (1.1.12) the sum is $4\pi\,\varepsilon_\lambda^{\alpha*}\varepsilon_\lambda^{\beta}W_{\alpha\beta}$, so $\sigma_\lambda = (4\pi^2\alpha/MK)\,\varepsilon_\lambda^{\alpha*}\varepsilon_\lambda^{\beta}W_{\alpha\beta}$. For a real photon K is its energy. For a virtual photon the usual convention takes K to be the energy of the real photon that would produce the same W: $W^2 = M^2 + 2MK$, so $MK = \frac{1}{2}(W^2 - M^2) = \nu(1-x)$.

D7. The elastic limit and the Rosenbluth formula

Derives (1.1.27), (1.1.28)

The final state. For X a single proton of momentum P', the sum over X in Eq. (1.1.12) is $\int d^3P'/\big((2\pi)^3 2E_{P'}\big)$ and a spin sum. The momentum integral with the four-dimensional delta function leaves $2\pi\,\delta\big((P+q)^2 - M^2\big)$, and $(P+q)^2 - M^2 = 2\nu - Q^2 = 2\nu(1-x)$. So

$$W_{\rm el}^{\alpha\beta} = \frac{1}{4\pi}\cdot 2\pi\cdot\frac{\delta(1-x)}{2\nu}\,T^{\alpha\beta} = \frac{\delta(1-x)}{4\nu}\,T^{\alpha\beta}, \qquad T^{\alpha\beta} = \frac{1}{2}\sum_{\rm spins}\langle P|j^{\beta\dagger}|P'\rangle\langle P'|j^{\alpha}|P\rangle.$$

A point particle. For a structureless spin-$\frac{1}{2}$ particle of unit charge, $\langle P'|j^\alpha|P\rangle = \bar{u}(P')\gamma^\alpha u(P)$. The trace is the one in D2 with masses, $\sum u\bar{u} = \not{P} + M$:

$$T^{\alpha\beta} = \frac{1}{2}\,\mathrm{Tr}\big[(\not{P}' + M)\gamma^\alpha(\not{P} + M)\gamma^\beta\big] = 2\big[P'^\alpha P^\beta + P'^\beta P^\alpha - g^{\alpha\beta}\,(P\cdot P' - M^2)\big].$$

With $P' = P + q$, $P\cdot P' - M^2 = \nu$. The coefficient of $-g^{\alpha\beta}$ in $W^{\alpha\beta}$ is $2\nu/4\nu = \frac{1}{2}$, and the coefficient of $P^\alpha P^\beta$ is $4/4\nu = 1/\nu$. The terms containing q are then fixed by current conservation. Comparing with Eq. (1.1.15): $F_1 = \frac{1}{2}\delta(1-x)$ and $F_2 = \delta(1-x)$.

The proton. The most general matrix element of a conserved vector current between spin-$\frac{1}{2}$ states has two terms, $\bar{u}(P')\big[F_D\,\gamma^\alpha + (i/2M)\,F_P\,\sigma^{\alpha\beta}q_\beta\big]u(P)$, with two form factors that depend on Q^2. The electric and magnetic combinations are $G_E = F_D - \tau F_P$ and $G_M = F_D + F_P$. Repeating the trace with this vertex gives Eq. (1.1.27). The algebra is longer and is not reproduced here; the check script evaluates the trace with explicit Dirac matrices for two choices of (G_E, G_M) and reproduces Eq. (1.1.27) to one part in 10^9.

Integrating over the peak. Put Eq. (1.1.27) into Eq. (1.1.22) and integrate over E' at fixed angle. From Eq. (1.1.8), $x \propto E'/(E-E')$ at fixed θ, so

$$\frac{d\ln x}{dE'} = \frac{1}{E'} + \frac{1}{E-E'} = \frac{E}{E'(E-E')}, \qquad \int dE'\,\delta(1-x)\,f(E') = \frac{E'(E-E')}{E}\,f(E')\Big|_{x=1}.$$

The F_2 term has $f = 1/(E-E')$ and gives E'/E times the form-factor combination. The F_1 term gives $(G_M^2/M)\tan^2(\theta/2)\cdot E'(E-E')/E$, and at $x=1$, $(E-E')/M = Q^2/2M^2 = 2\tau$. The sum is Eq. (1.1.28). The factor E'/E is the recoil of the proton, with $E' = E/\big(1+(2E/M)\sin^2(\theta/2)\big)$ on the peak.

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